Tridendriform structures on faces of hypergraph associahedra

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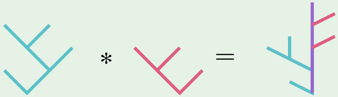
Outline

Prologue

Outline

Tridendriform algebras [Loday-Ronco, 2004; Chapoton, 2002]

Example



Tridendriform algebras [Loday-Ronco, 2004; Chapoton, 2002]

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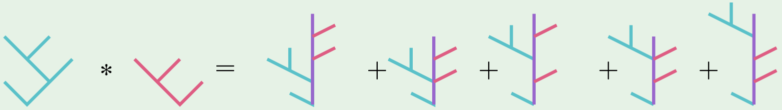
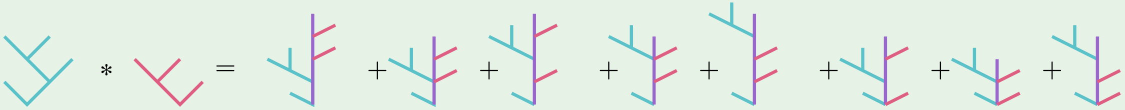


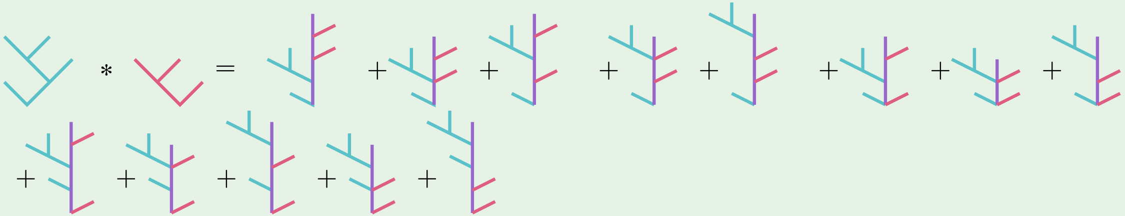
Tridendriform algebras [Loday-Ronco, 2004; Chapoton, 2002]

Example



Tridendriform algebras [Loday-Ronco, 2004; Chapoton, 2002]

Example



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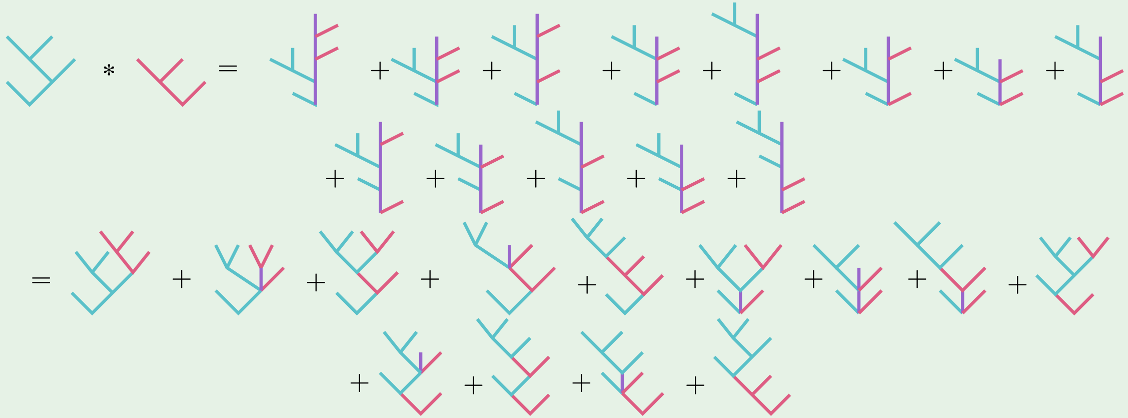

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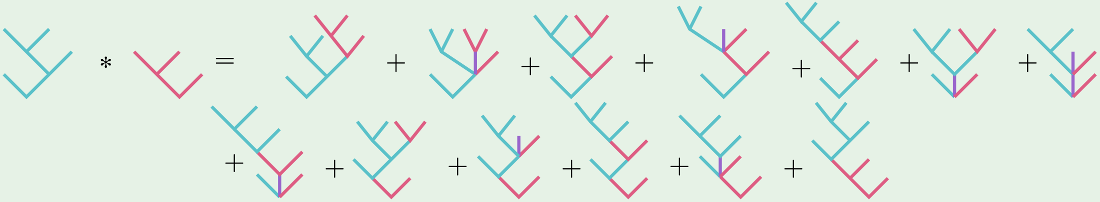
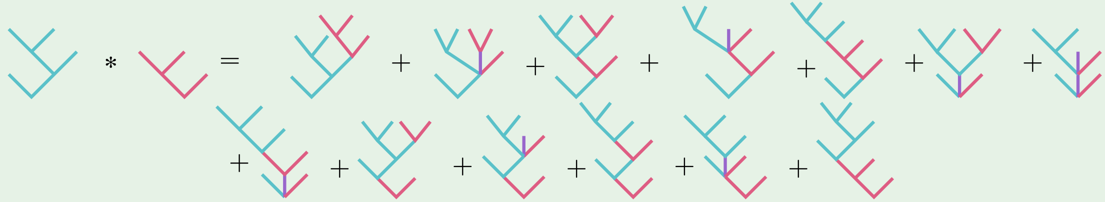


Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

Tridendriform algebras [Loday-Ronco, 2004; Chapoton, 2002]

Example



Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

Idea:

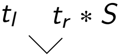
Three kinds of trees (looking at the root) : why not splitting in three the product $*$?

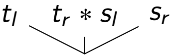
Inductive definition of tridendriform products on trees

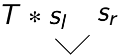
If $T =$ 

and $S =$ 

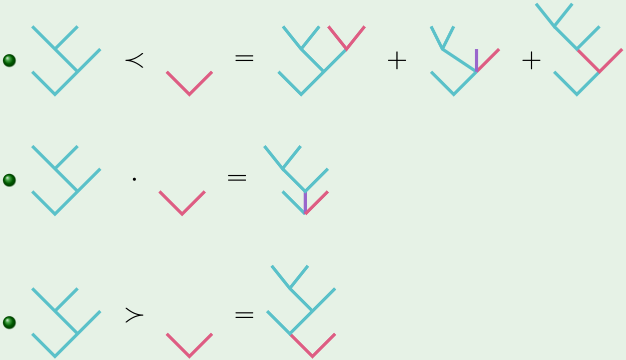
,

$T \prec S =$ 

$T \cdot S =$ 

and $T \succ S =$ 

Examples :



Tridendriform algebras

Definition (Loday, Ronco, 2004 ; Chapoton 2002)

A tridendriform algebra is a vector space A endowed with products $\prec: A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ: A \otimes A \to A$, such that:

1. $(a \prec b) \prec c = a \prec (b * c)$,
2. $(a * b) \succ c = a \succ (b \succ c)$,
3. $(a \succ b) \prec c = a \succ (b \prec c)$,
4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
5. $(a \succ b) \cdot c = a \succ (b \cdot c)$,
6. $(a \prec b) \cdot c = a \cdot (b \succ c)$,
7. $(a \cdot b) \prec c = a \cdot (b \prec c)$,

with $* = \prec + \cdot + \succ$

Algebra on packed words WQSym [Duchamp-Hivert-Novelli-Thibon, 2011]

$$u\#v = \sum_{\substack{\text{pack}(\alpha)=u \\ \text{pack}(\beta)=v \\ c_\#}} \alpha\beta,$$

where $c_\# = \min(\alpha) < \min(\beta)$ for $\# = \prec$,
$c_\# = \min(\alpha) = \min(\beta)$ for $\# = \cdot$,
and $c_\# = \min(\alpha) > \min(\beta)$ for $\# = \succ$.

Example :

$$11 \succ 221 = 22221 + 33221 + 22331$$

$$11 \cdot 221 = 11221$$

$$11 \prec 221 = 11332$$

Algebra on packed words WQSym [Duchamp-Hivert-Novelli-Thibon, 2011]

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Example :

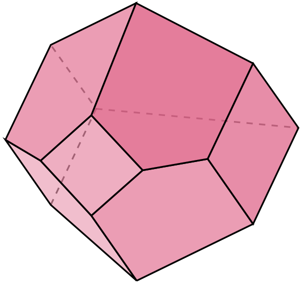
$$11 \succ 221 = 22221 + 33221 + 22331$$

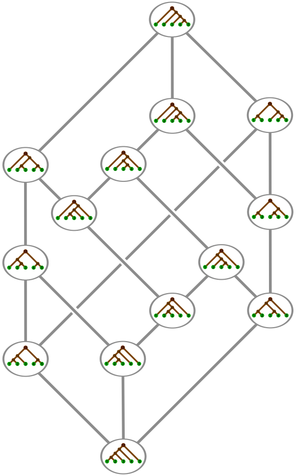
$$11 \cdot 221 = 11221$$

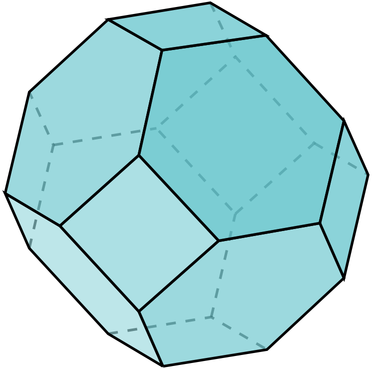
$$11 \prec 221 = 11332$$

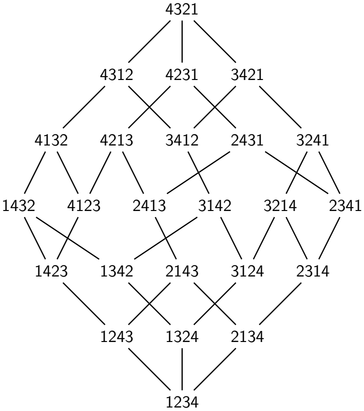
Tridendriform products $\Rightarrow$ WQSym free tridendriform algebra on infinitely many generators [Vong, Burgunder-Curien-Ronco, 2015]

Link with associahedra and permutohedra



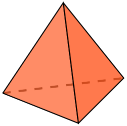




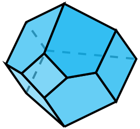

4321
4312 4231 3421
4132 4213 3412 2431 3241
1432 4123 2413 3142 3214 2341
1423 1342 2143 3124 2314
1243 1324 2134
1234

Hypergraph polytopes (a.k.a. nestoedra)

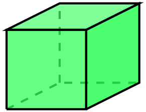
Outline



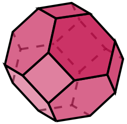
Simplices



Associahedra



Hypercubes



Permutohedra

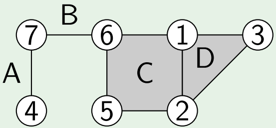
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

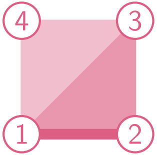
- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

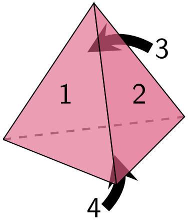
Example of an hypergraph on $[1; 7]$

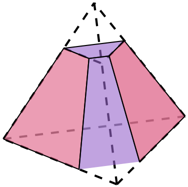


.

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])







Constructs [Postnikov; Curien-Ivanovic-Obradović]

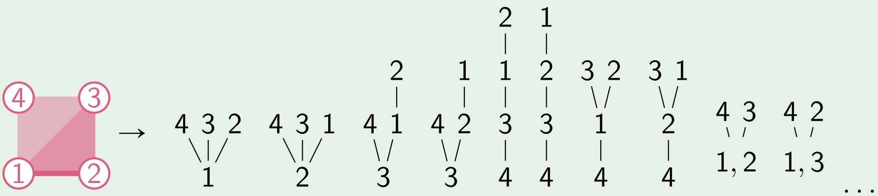
Constructs

A construct of a hypergraph H is defined inductively. For $E \subset V(H)$ (the set of vertices of H),

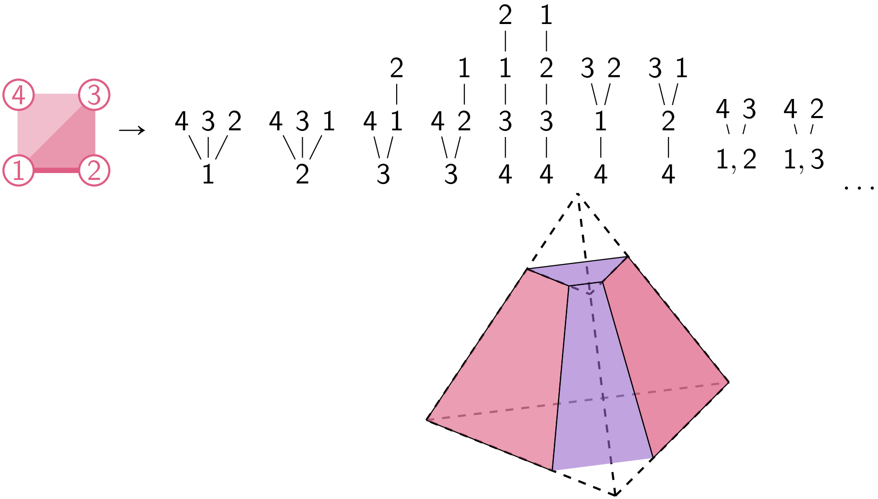
- If $E = V(H)$, the construct is the rooted tree with only one node labelled by E,
- Otherwise, denoting by $(T_1, \ldots, T_n)$ constructs on every connected component in $H - E$, a construct of H can be obtained by grafting these trees on a node labelled by E.

The set of constructs of a given hypergraph labels faces of the associated polytope.

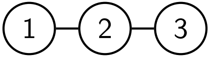
First example:



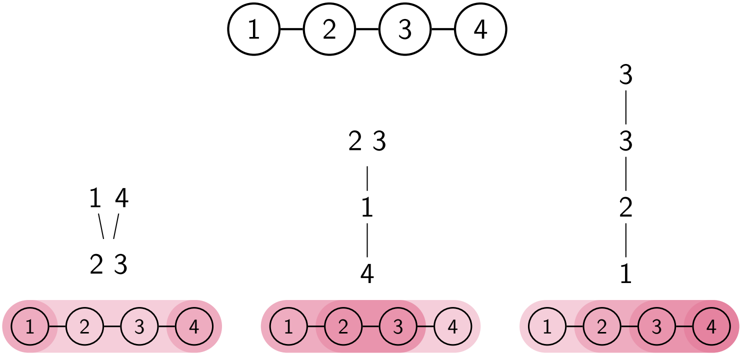
First example geometrically



Let's practice



Correspondence Tubings = Constructs = Spines

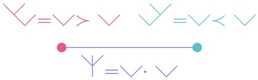


Combinatorial interpretation of constructs

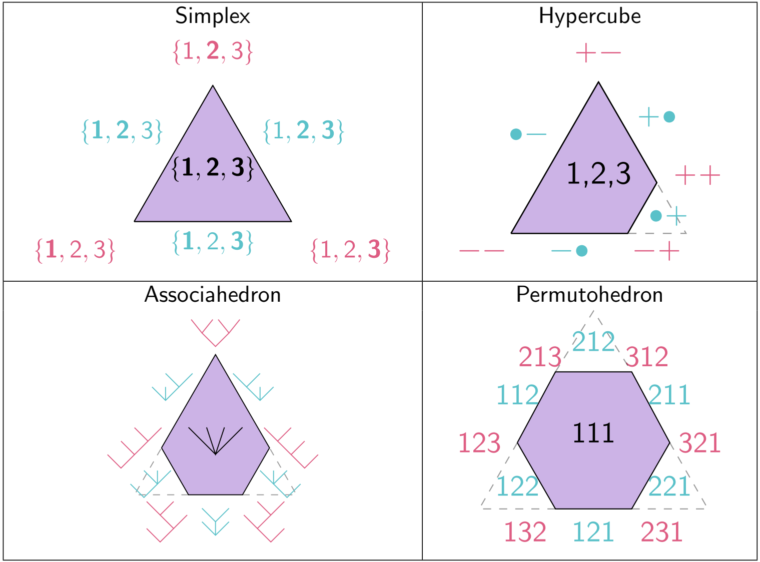
Simplex To a k-dimensional face $\{a_1, \ldots, a_k\}$ is associated the multipointed set $(V(H), \{a_1, \ldots, a_k\})$

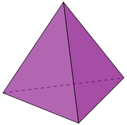
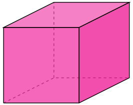
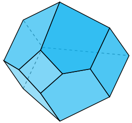
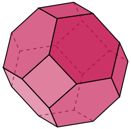
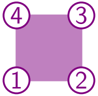
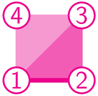
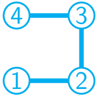
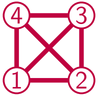
Cube To a k-dimensional face is associated the set of words of length $n-1$ on $+$, $-$ and $\bullet$ with k $\bullet$ (or left-comb trees)

Associahedron To a k-dimensional faceis associated a planar tree on $n-k$ nodes.



Permutohedron To a k-dimensional face is associated a surjection of height k, i.e., a packed word on $\{1, \ldots, n-k\}$


Simplex
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
Hypercube
+−
•−
+•
1,2,3
++
•+
−−
−•
−+
Associahedron
Permutohedron
212
213
312
112
211
111
123
321
122
221
132
121
231

Polytope	Simplex	Hypercube	Associahedron	Permutohedron
Photo				
Associated hypergraph				
Combinatorial objects	multipointed sets	left-comb tree	planar trees	packed words
Cardinality	$2^{n+1} - 1$ (A074909)	3^n (A013609)	Super-Catalan (A001003)	Fubini nbrs (A000670)

Algebraic structures on hypergraph polytopes

Outline

Heuristics for a tridendriform structure

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

- $S \prec T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,**
- $S \succ T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** B**,**
- $S \cdot T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** $A \cup B$**.**

Tridendriform products defined on faces of simplices [Loday-Ronco, Chapoton]

On simplices, we get the following (triass) products, denoting by $(\mathcal{X}, A)$ the multipointed set whose underlying set is $\mathcal{X}$ and whose set of pointed elements is A:

$$
\begin{aligned}
(\mathcal{X}, A) \prec (\mathcal{Y}, B) &= (\mathcal{X} \cup \mathcal{Y}, A) \\
(\mathcal{X}, A) \succ (\mathcal{Y}, B) &= (\mathcal{X} \cup \mathcal{Y}, B) \\
(\mathcal{X}, A) \cdot (\mathcal{Y}, B) &= (\mathcal{X} \cup \mathcal{Y}, A \cup B)
\end{aligned}
$$

Tridendriform products defined on faces of hypercubes

Applying this construction to hypercube gives :

$$
\begin{aligned}
u \prec v &= u\,(-^{|v|}) \\
u \succ (v_1 \; + \; v_2) &= \begin{cases} (u \star v_1) \; + \; v_2 & (v_1 \neq \epsilon) \\ u + v_2 & (v_1 = \epsilon) \end{cases} \\
u \cdot (v_1 \; + \; v_2) &= u\,(-^{|v_1|}) \bullet v_2
\end{aligned}
$$

where each word begins by a $+$ and the $+$ denotes the rightmost occurence of $+$.

Question

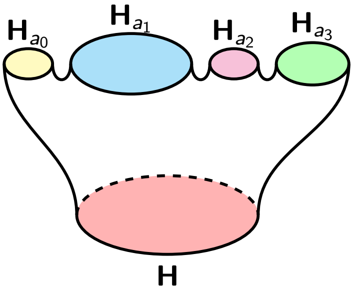
How to formalize this construction ?

Universe and preteam

The considered hypergraphs belong to a set of hypergraphs $\mathfrak{U}$, called universe.

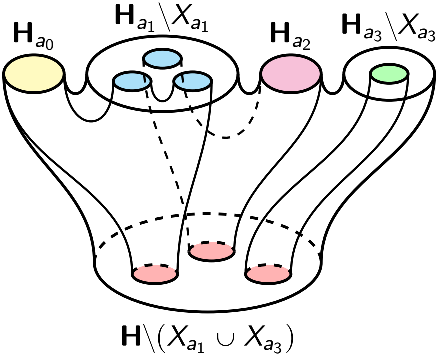
A preteam is a pair $\tau = (\{\mathbf{H}_a | a \in A\}, \mathbf{H})$ where

- $\{\mathbf{H}_a | a \in A, \mathbf{H}_a \in \mathfrak{U}\}$ is a set of pairwise disjoint hypergraphs, called participating hypergraphs
- $\mathbf{H} \in \mathfrak{U}$ is a hypergraph such that $H = \bigcup_{a \in A} H_a$, called supporting hypergraph.



Strict and semi-strict teams

A preteam is a (resp. semi-strict) strict team if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are in $\mathfrak{U}$ and included in the connected components of $\mathbf{H} \setminus \left(\bigcup_{a \in A} X_a\right)$ (resp. or totally disconnected)



$(X_{a_0} = X_{a_2} = \varnothing)$

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Examples:

- Simplices
- Hypercubes
- Associahedra
- Permutohedra

Product

Considering a team E and denoting by δ a tuple of constructs of the team's participating hypergraphs, we inductively associate to δ a sum of constructs of the supporting hypergraph:

$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} (\bigcup_{b \in B} X_b)(*(\delta_1^B), \dots, *(\delta_{n_B}^B)), \quad (1)$$

Polydendriform structure

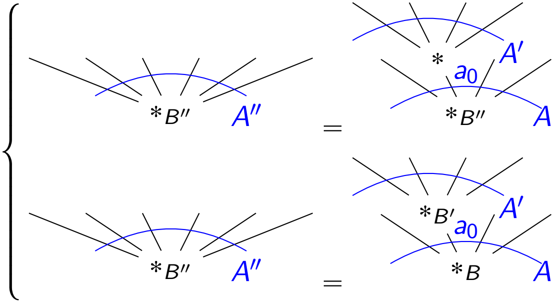
Let us introduce new operations

$$*_B(\delta) = (\bigcup_{b\in B} X_b)(*(\delta_1^B), \dots, *(\delta_{n_B}^B))$$

such that the product splits

$$*(\delta) = \sum_{\emptyset\subset B\subseteq A} q^{|B|-1} *_B(\delta)$$

It satisfies relations:

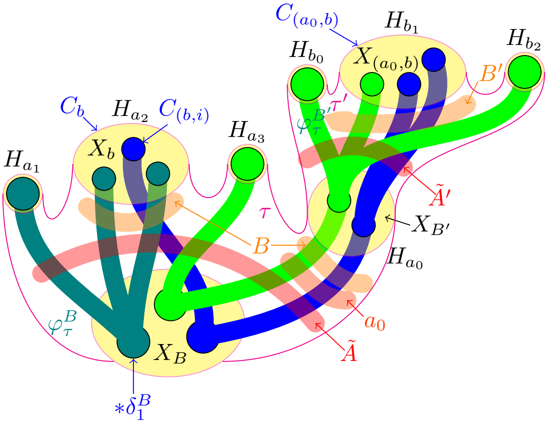


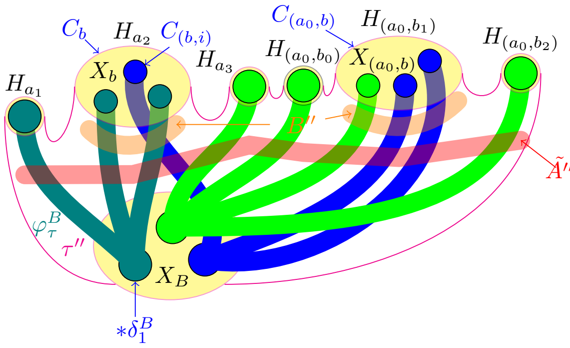
if $B'' \subseteq A\backslash\{a_0\}$

if $B'' \nsubseteq A\backslash\{a_0\}$

Associative clan

A set of (resp. semi-strict) strict team with "good" closure properties is called strict clan (each connected component obtained from the supporting hypergraph is itself a supporting hypergraph of a team).





Associativity of $*$

Theorem (Curien-D.O.-Obradović, 21+)

Consider a clan $\mathcal{C}$. The product $*$ is associative if

- *$\mathcal{C}$ is strict,*
- *or $\mathcal{C}$ is semi-strict and $q = -1$.*

- Strict clans: Associahedra, Permutohedra, Restrictohedra, ...
- Semi-strict clans: Simplices, Hypercubes, Cyclohedra, ...

Restrictohedra and associated examples

Outline

Restrictohedra

Given $\mathbf{K}$ a (possibly infinite) hypergraph.

$\mathbf{K}$-restrictohedron:

$$\mathfrak{U}_{\mathbf{K}} = \{\mathbf{K}_X | X \subseteq K, X \text{ finite and } \mathbf{K}_X \text{ connected}\}$$

Theorem (Curien-D.O.-Obradović, 21+)

The clan

$$\Xi_{\mathbf{K}} = \{\{(\{\mathbf{K}_{X_a} | a \in A\}, \mathbf{K}_X)\} | X_a \textit{ partition of } X \textit{ and } \mathbf{K}_{X_a}, \mathbf{K}_X \in \mathfrak{U}_{\mathbf{K}}\}$$

is a strict associative clan.

Characterization of restrictohedra

Proposition (Curien-D.-O.-Obradović, 21+)

A universe $\mathfrak{U}$ is of the form $\mathfrak{U}_{\mathbf{K}}$, for some hypergraph $\mathbf{K}$, if and only if it satisfies the following four conditions:

(Hierarchy) If $\mathbf{H}, \mathbf{G} \in \mathfrak{U}$ and $G = H$, then $\mathbf{G} = \mathbf{H}$.

(Minimality) If $\mathbf{H} \in \mathfrak{U}$ and $e \in \mathbf{H}$, if $\mathbf{G} \in \mathfrak{U}$ is such that $e \subseteq G$, then $e \in \mathbf{G}$.

(Restriction) If $\mathbf{H} \in \mathfrak{U}$, and if $X \subseteq H$ is such that $\mathbf{H}_X$ is connected, then there exists $\mathbf{G} \in \mathfrak{U}$ such that $G = X$.

(Join) If $\mathbf{K}, \mathbf{L} \in \mathfrak{U}$ are such that $K \cap L$ is non-empty, then there exists $\mathbf{H} \in \mathfrak{U}$ such that $\mathbf{K}, \mathbf{L} \subset \mathbf{H}$.

Ordered universe and Atomicity

Consider $K \subseteq \mathbb{Z}$.

- A clan Ξ is called atomic if for any $\mathbf{H}$ in $\mathfrak{U}$, there exist $\mathbf{H}_1, \mathbf{H}_2 \in \mathfrak{U}$ such that $H_1 = H \backslash \{\min(H)\}$, $H_2 = H \backslash \{\max(H)\}$ and both $(\{\{\{\min(H)\}\}, \mathbf{H}_1\}, \mathbf{H})$ and $(\{\mathbf{H}_2, \{\{\max(H)\}\}\}, \mathbf{H})$ are in Ξ.
- A hypergraph $\mathbf{K}$ is graph-like if it has the same connected sets as its restriction to the graph formed by its vertices and edges.

Proposition

The universe $\mathfrak{U}_{\mathbf{K}}$ is atomic if and only if $\mathbf{K}$ is graph-like, and if, for all $a < b < c \in K$, if $\{b, c\} \in \mathbf{K}$ and $\{a, c\} \in \mathbf{K}$, then $\{a, b\} \in \mathbf{K}$, and if $\{a, b\} \in \mathbf{K}$ and $\{a, c\} \in \mathbf{K}$, then $\{b, c\} \in \mathbf{K}$.

Translatohedra

For $X \subseteq \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\mathrm{Sh}(X, k) = \{x + k \mid x \in X\}$$

$$\mathrm{Sh}(\mathbf{H}, k) = \{\mathrm{Sh}(e, k) | e \in \mathbf{H}\}$$

A universe $\mathfrak{U}$ is closed under translation if, whenever a hypergraph $\mathbf{H}$ belongs to $\mathfrak{U}$, then so does $\mathrm{Sh}(\mathbf{H}, k)$ for all k.

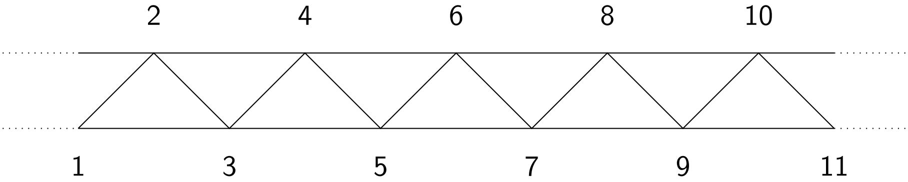
Proposition

Let A be a non-empty subset of strictly positive integers that is closed under truncated subtraction (i.e. if $a, b \in A$ and $a < b$, then $b - a \in A$), and let $\mathbf{G}$ be the graph defined from A as follows:

$$\mathbf{G} = \{\{\ell\} | \ell \in \mathbb{Z}\} \cup \{\mathrm{Sh}(\{0, a\}, \ell) | a \in A, \ell \in \mathbb{Z}\}.$$

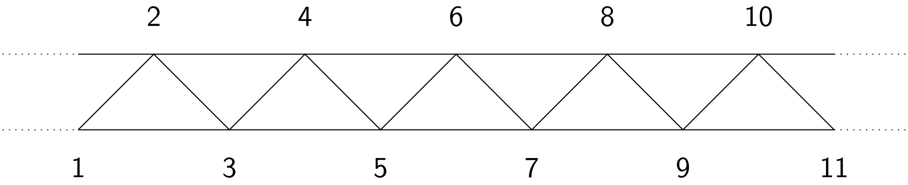
Then $\mathfrak{U}_{\mathbf{G}}$ is atomic and closed by translation. Moreover, every restrictohedron that is atomic and closed by translation is equivalent to a restrictohedron obtained in this way.

Friezohedra



- strict associative clan
- atomic
- closed by translation
- between associahedra and permutohedra

Friezohedra



n=4

Compactohedra

Definition

A compactohedron $\mathfrak{U}_{\mathbf{K}}$ is a restrictohedron whose underlying hypergraph over $\mathbb{Z}$, $\mathbf{K}$, satisfies:
(compactness) If there exists $e = \{a_1, \ldots, a_k\}$ in $\mathbf{K}$, then for any increasing map $f : \{a_1, \ldots a_k\} \to \mathbb{Z}$ such that $|f(a_i) - f(a_j)| \leqslant |a_i - a_j|$ for any i and j, we have $f(e) = \{f(a_1), \ldots, f(a_k)\} \in \mathbf{K}$.

Proposition

Let k be either a positive integer or $+\infty$ and let $\mathbf{G}^k$ be the graph defined as follows:

$$\mathbf{G}^k = \{\{\ell\} | \ell \in \mathbb{Z}\} \cup \{\mathrm{Sh}(\{0, p\}, \ell) | 1 \leqslant p \leqslant k, \ell \in \mathbb{Z}\}.$$

Then $\mathfrak{U}_{\mathbf{G}^k}$ is an atomic compactohedron. Moreover, every atomic compactohedron is equivalent to a restrictohedron obtained in this way.

What about the coproduct ?

Try:

$$\Delta(S) = \sum_{\text{c coupe}} R_c(S) \otimes *(F_c(S)) + 1 \otimes S$$

Problem: renormalisation

Conjecture

Only possible for associahedra and permutohedra

Conclusion and research directions

Get a general frame to define an associative product and tridendriform operations.

Research directions

- Are the resulting tridendriform algebras free ?
- Study of the polydendriform operad
- Operads on polytopes (avec E. Burgunder (IMT, Toulouse) and P.-L. Curien (IRIF))
- Link with generalized Tamari order (avec P.-L. Curien (IRIF) et J. Obradović (Serbie))

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Thank you !