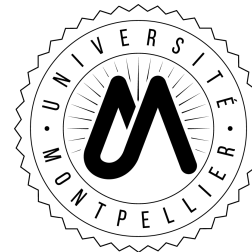


Posets, incidence Hopf algebras and operads

Part IV

Bérénice Delcroix-Oger



Journées du GDR Renorm
Du 14 au 18 novembre 2022, Calais

Outline

- 1 Posets and incidence Hopf algebra
- 2 Hypertrees
- 3 Operads and homology
- 4 Back to the cohomology of the hypertree posets
joint work with Clément Dupont (IMAG)

Recall from yesterday

Warning !

Today, we will consider the posets of hypertrees on $\{0, \dots, n\}$, denoted by HT_n with a slight abuse of the notation !

Proposition (McCullough–Miller, 1996)

HT_n is Cohen-Macaulay.

Proposition

The Möbius number of HT_n is given by:

$$\mu(\text{HT}_n) = (-1)^n \frac{(2n-1)!}{n!}$$

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying vector space $\text{PostLie}(V)$ of **post-Lie** operad is spanned by Lie brackets of planar trees with nodes labeled by V . The **substitution** of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).

Proposition

*The vector space of n -ary operations of **Post-Lie operad** has dimension*

$$\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}.$$

The post-Lie products \triangleleft and $\{ ; \}$ satisfy

- $\{ ; \}$ is a Lie bracket

$$(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) - (x \triangleleft z) \triangleleft y + x \triangleleft (z \triangleleft y) = x \triangleleft \{y, z\}$$

$$\{x, y\} \triangleleft z = \{x \triangleleft z, y\} + \{x, y \triangleleft z\}$$

Back to the cohomology of the hypertree posets
joint work with Clément Dupont (IMAG)

Outline

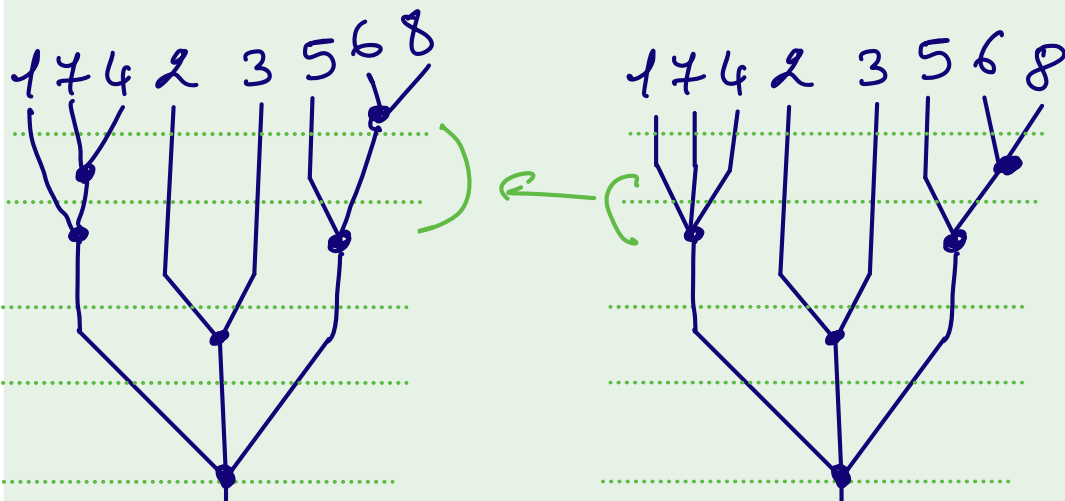
- 1 Posets and incidence Hopf algebra
- 2 Hypertrees
- 3 Operads and homology
- 4 Back to the cohomology of the hypertree posets
joint work with Clément Dupont (IMAG)

Motivation : Partition posets and Lie operad

$$C_j(\Pi_n) = \mathbb{C} \cdot \{ \hat{0}_{\Pi_n} = \pi_{-1} < \dots < \pi_{j+1} = \hat{1}_{\Pi_n} \mid \pi_l \in \Pi_n, \forall l \in [-1; j+1] \}$$

Example : leveled cobar construction

$$\begin{aligned} & \langle 1, \dots, 8 \rangle < \langle 1, 7, 4 \rangle \langle 2, 3 \rangle \langle 5, 6, 8 \rangle \\ & < \langle 1, 7, 4 \rangle \langle 2 \rangle \langle 3 \rangle \langle 5, 6, 8 \rangle < \\ & \Leftarrow \langle 6, 8 \rangle \langle 1 \rangle \langle 2, \dots, 7 \rangle < \langle 1 \rangle \langle 2 \rangle \dots \langle 8 \rangle \end{aligned}$$



$\langle 1, 7, 4 \rangle \langle 2, 3 \rangle \langle 5, 6, 8 \rangle$

Theorem (Fresse, 04)

The action of the symmetric group on the cohomology of the partition posets Π_n is given by

$$\tilde{H}_{n-1}(\Pi_n) = \text{Lie}(n)^\vee \otimes \text{sgn}_n$$

where $\text{Lie}(n)^\vee$ is the dual module of Lie .

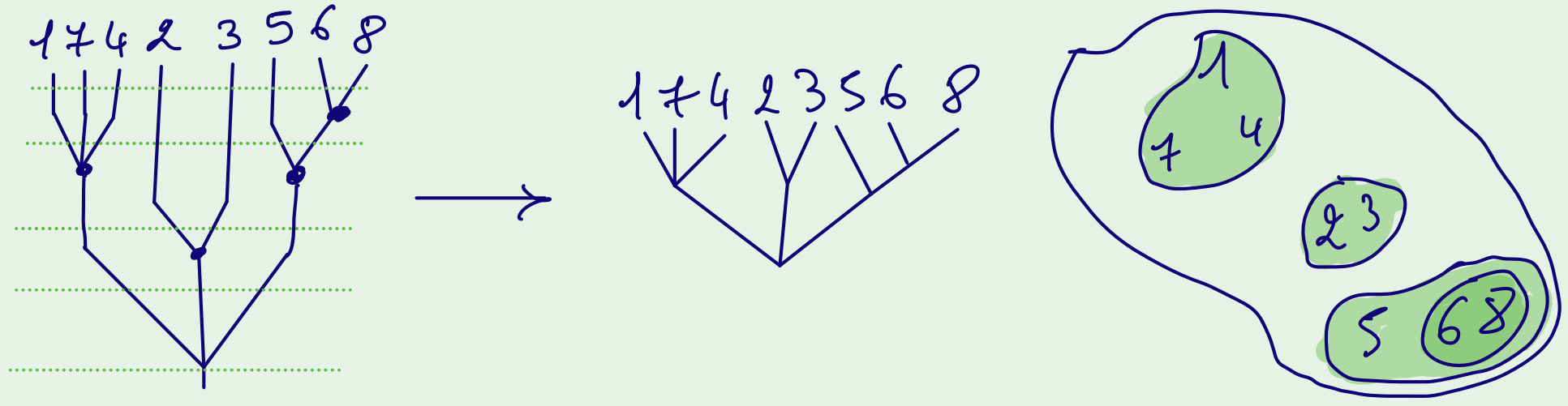
To nested sets

Problem

There are no operadic structure on the leveled cobar construction, but there is one on the cobar construction !

Solution :

Forget about the levels !



This is what we obtain when we consider nested sets instead of chains !

Join and meet

Definition

The **join** of a subset S of a poset P is the supremum (least upper bound) of S , denoted by $\bigvee S$.

In the partition poset

$$\{1, 3\}\{2, 4, 6\}\{5\} \vee \{1, 2, 3\}\{4, 6\}\{5\} = \{1, 3\}\{2\}\{4, 6\}\{5\}.$$

More generally, the join of two partitions $p = \{p_1, \dots, p_k\}$ and $q = \{q_1, \dots, q_l\}$ is $\{p_i \cap q_j\}_{1 \leq i \leq k, 1 \leq j \leq l}$

Definition

The **meet** of a subset S of a poset P is the infimum (greatest lower bound) of S , denoted by $\bigwedge S$.

In the partition poset $\{1, 3\}\{2, 4, 6\}\{5\}, \{1, 2, 3\}\{4, 6\}\{5\} = \{1, 2, 3, 4, 6\}\{5\}$

The meet of two partitions p and q is obtained by merging every parts p_1 and p_2 of p such that there is a part in q intersecting non trivially both.

Lattices

Definition

A **join-semilattice** is a poset that has a join for any nonempty finite subset.

The partition posets are join-semilattices.

Lemma

The cartesian product of join-semilattices is a join-semilattice.

Lemma

$$HT_n^a = \prod_{v \in V(a)} \prod_{\text{deg}(v)}$$

Proposition

Every maximal interval HT_n^a in the hypertree posets is a join-semilattice.

Let us recall from the first lecture

Let $\pi \in \Pi_n$, $\pi = \{V_1, \dots, V_k\}$

Lemma

The following isomorphisms hold:

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^k \Pi_{|V_i|} \quad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

Building sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider \mathcal{L} a finite join-semilattice. For any $S \subseteq \mathcal{L}$ and $x \in \mathcal{L}$, we write

$$S_{\geq x} = \{y \in S \mid y \geq x\}.$$

Definition

A **building set** is a subset \mathcal{G} in $\mathcal{L}_{< \hat{1}}$ such that for any $x \in \mathcal{L}_{< \hat{1}}$ and $\max \mathcal{G}_{\geq x} = \{g_1, \dots, g_k\}$, there is an isomorphism of posets

$$[x, \hat{1}] \simeq \prod_{i=1}^k [g_i, \hat{1}].$$

Example on partition posets:

The minimal building set for partition posets is the set of partitions whose parts are all trivial but one.

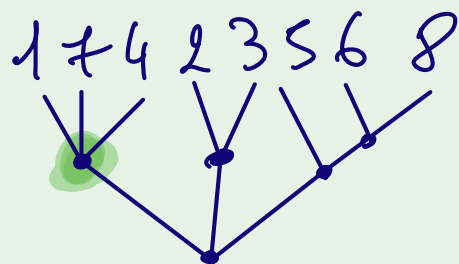
Nested sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider a join-semilattice \mathcal{L} and an associated building set \mathcal{G} .

Definition

A **nested set** is a subset S of \mathcal{G} such that for any set of incomparable elements x_1, \dots, x_t in S ($t \geq 2$), the meet $x_1 \wedge \dots \wedge x_t$ exists and does not belong to \mathcal{G} .

Example on partition posets:



$\nwarrow \pi_p$ non trivial

$$p, q \in S \Rightarrow \pi_p \leq \pi_q \text{ or } \pi_q \leq \pi_p \text{ or } \pi_p \wedge \pi_q = \emptyset$$

$\nearrow \pi_q$ non trivial

$$\left\{ \begin{array}{l} \{1, 7, 4\}, \{2, 3\}, \{5, 6, 8\}, \{1, \dots, 8\} \\ \{2, 3\}, \{1, 7, 4\}, \{5, 6, 8\}, \{1, \dots, 8\} \\ \{5, 6, 8\}, \{1, 7, 4\}, \{2, 3\}, \{1, \dots, 8\} \\ \{6, 8\}, \{1, 7, 4\}, \{2, 3\}, \{5, 6, 8\} \end{array} \right.$$

Topological result

The \mathcal{G} -nested sets form an abstract simplicial complex, called the **nested set complex**.

Proposition (Feichtner–Müller, 05)

*Consider a join-semilattice \mathcal{L} and an associated building set \mathcal{G} . The associated nested set complex is **homotopy equivalent** to the order complex of the poset.*

For partition posets

The cobar resolution (for the Commutative operad) corresponds to the cochain complex of the nested set complex associated with the minimal building set.

Let us adapt building sets to hypertree posets

Definition

A **building set** of HT_n is a subset \mathcal{G} in $HT_n - \max(HT_n)$ such that for any $x \in HT_n$ and $\max \mathcal{G}_{\geq x} = \{g_1, \dots, g_k\}$, there is an isomorphism of posets for every tree a in HT_n :

$$[x, a] \simeq \prod_{i=1}^k [g_i, a].$$

Example on hypertree posets:

The minimal building set for hypertree posets is the set of hypertrees whose edges are of size two but one.

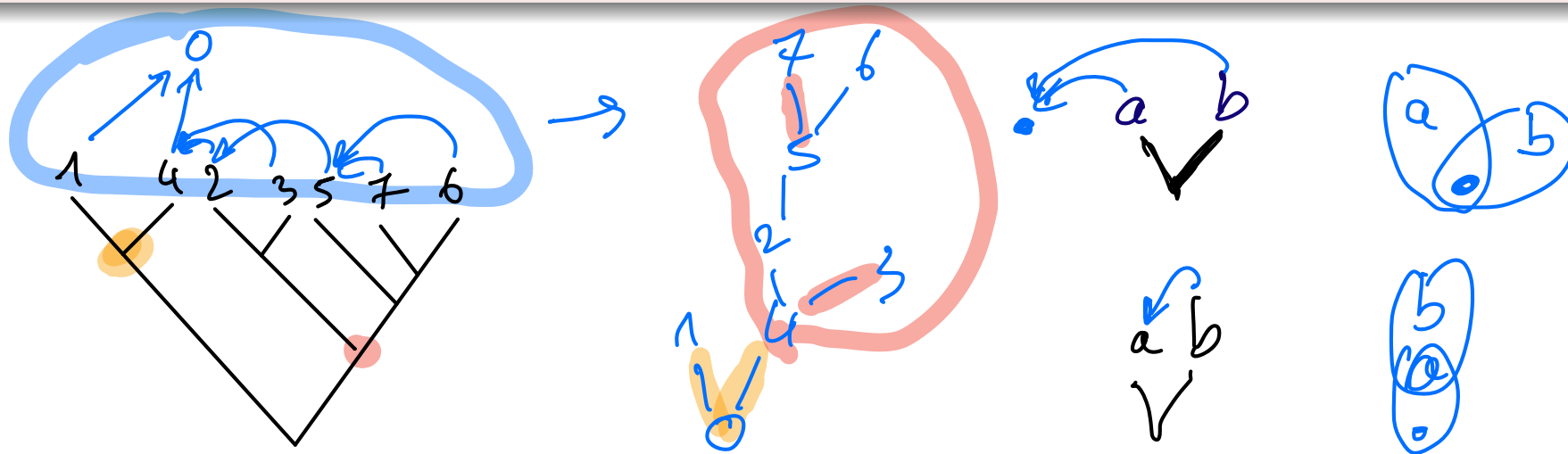
The nested set complex of hypertrees

The nested sets of hypertrees are the following combinatorial objects:

Definition

A bitree is a pair (T, τ) of trees such that

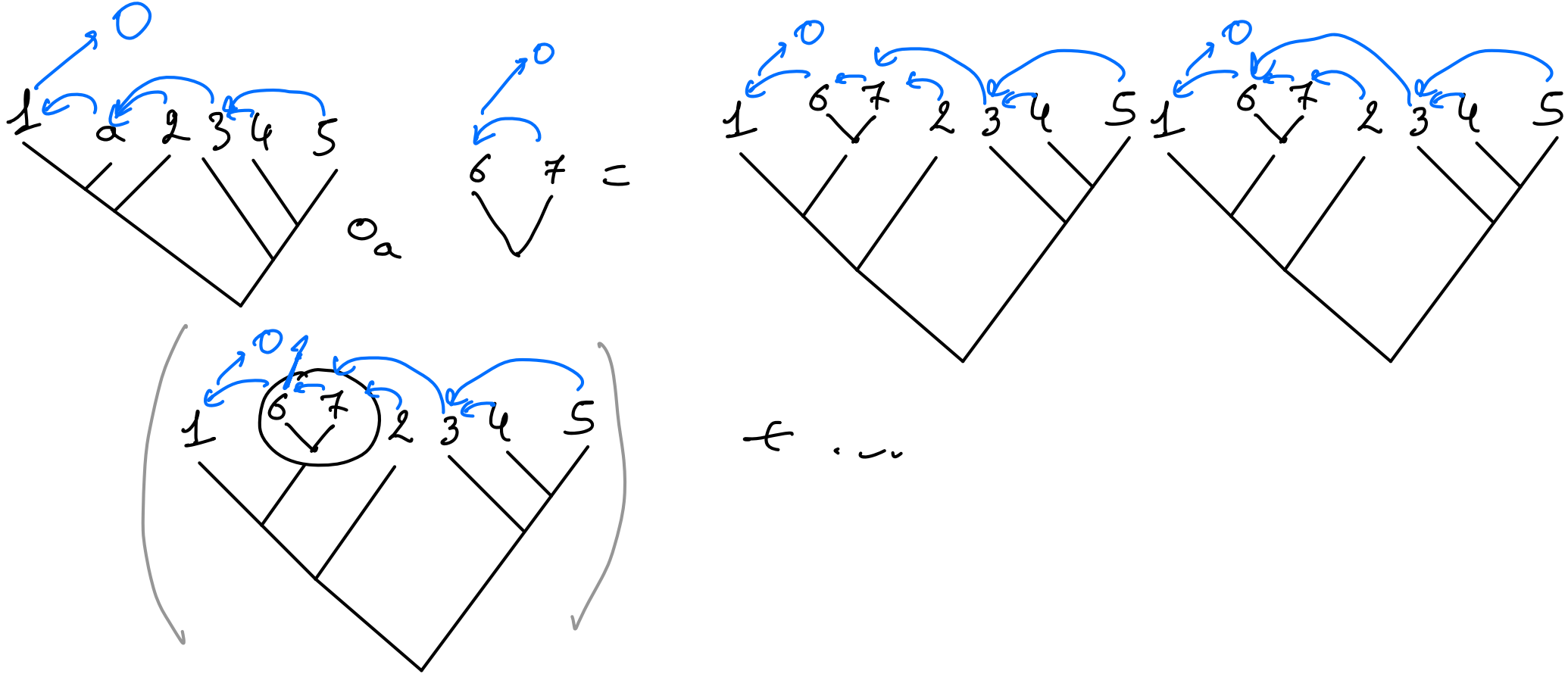
- T is a (non planar) rooted reduced (no vertex of valency 2) tree with leaves labeled by $\{1, \dots, n\}$
- τ is a (non planar oriented) tree whose vertices are labeled by $\{0, \dots, n\}$ and whose root is 0
- for any internal vertex s in T , the restriction of τ to edges leaving the leaves above s is connected



Operadic composition \rightarrow root tree ∞ ?

The operadic composition of a bitree b in a node v is as follows:

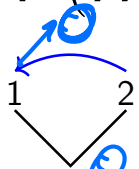
- the blue children of v are grafted to some nodes in b (pre-Lie type magmatic composition)
- the bottom tree of b is grafted at the place of the leaf v (usual magmatic composition)

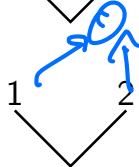


Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

Let us consider the map

$$\text{Post-Lie} \xrightarrow{\phi} H^\bullet(\text{HT}_\bullet)$$

$1 \triangleleft 2 \mapsto$


$\{1; 2\} \mapsto$


Theorem (DO–Dupont, 22+)

The map ϕ is an operad morphism. The cohomology of the hypertree poset can be endowed with an operadic structure. It is then isomorphic to the suspension of post-Lie operad.

Idea of the proof : Comb and twist

First, we choose a planar representation of the bitree with blue arrows to the left. The proof relies on two lemmas:

Lemma (Comb)

In the cohomology, any bitree is equivalent to the bitree obtained by exchanging the bottom tree with a comb-shaped tree

Lemma (Twist)

In the cohomology, any nodes which are not close relatives (parent-child or siblings) can be exchanged

We then prove that ϕ is a morphism of operad and is surjective.

Where does the idea of the composition come from ?

Let us consider the map $a : \text{HT}_n^0 \rightarrow \Pi_n$. We define

$$(\text{HT})_{\leq \pi} := a^{-1}(\Pi_{\leq \pi}) \quad \text{and} \quad (\text{HT})_{\geq \pi} := a^{-1}(\Pi_{\geq \pi}).$$

Define the maps

$$\varphi : (\text{HT})_{\leq \pi} \rightarrow \text{HT}(\pi)$$

and

$$\psi : (\text{HT})_{\geq \pi} \rightarrow \prod_{t \in \pi} \text{HT}(t)$$

obtained respectively by contracting parts of π to an element and splitting the hypertree according to the parts of π .

The idea is to use these maps to define a composition:

$$\begin{aligned} C^\bullet(\text{HT}(\pi)) \otimes \bigotimes_{T \in \pi} C^\bullet(\text{HT}(T)) &\simeq C^\bullet(\text{HT}(\pi)) \otimes C^\bullet\left(\prod_{T \in \pi} \text{HT}(T)\right) \\ &\xrightarrow{\phi^* \otimes \psi^*} C^\bullet(\text{HT}_{\leq \pi}) \otimes C^\bullet(\text{HT}_{\geq \pi}) \rightarrow C^\bullet(\text{HT}_n) \end{aligned}$$

What about pre-Lie ?

Work in progress

- We hope to recover pre-Lie as a left post-Lie module.
- Hypertree poset has the same cohomology as the poset of partitions decorated by Operad ComTrias [Vallette, 07] : both posets are two projections of the poset of multi-pointed hypertrees (whose cohomology group is post-Lie operad).
- This fits in a more general setting, which applies to examples of families of poset (F_n) endowed with a "good" projection $a : F_n \rightarrow \Pi_n$.

References

- Koszul duality of operads and homology of partition posets, B. Fresse, *"Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory"*, *Contemp. Math.* 346, Amer. Math. Soc., 115-215 (2004)
- On the topology of nested set complexes, E. Feichtner–I. Müller, *Proc. Am. Math. Soc.* 133, No. 4, 999-1006(2005)

Thank you very much for your attention to this four lectures !

as I have checked carefully with the fake mistakes in these slides !