A Bialgebra on Hypertree and Partition Posets

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Sommaire

Partition Posets

For n, a positive integer,

Definition

The Partition poset (partially ordered set) on n elements Π_n is the poset whose objects are partitions of n ordered by :

$$p_1 \leq p_2 \iff \text{each part of } p_1 \text{ is the union of some parts of } p_2$$

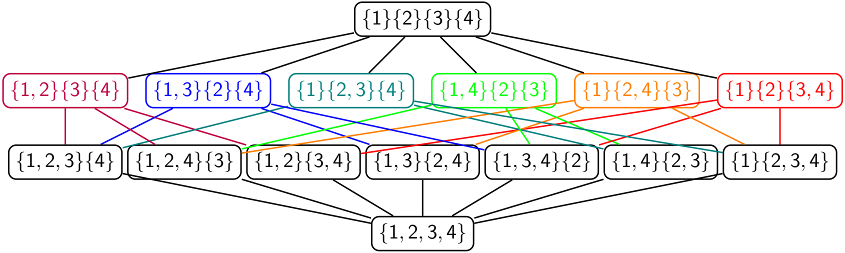


Figure: The poset Π_4

Incidence Hopf Algebra

Definition

The direct product of two posets $(P_1, \leq_{P_1})$ *and* $(P_2, \leq_{P_2})$ *is the set* $P_1 \times P_2$ *endowed with the following partial order :*

$$(x_1, x_2) \leq_{P_1 \times P_2} (y_1, y_2) \Longleftrightarrow x_i \leq_{P_i} y_i \forall i \in \{1, 2\}$$

We consider posets up to isomorphisms of posets.
Let us consider a family of bounded posets $\mathcal{F}$

- Interval closed : $\forall P \in \mathcal{F}, [a, b] \subseteq P \Rightarrow [a, b] \in \mathcal{F}$,
- and Stable under direct product : $\forall P, Q \in \mathcal{F}, P \times Q \in \mathcal{F}$.

We endow the $\mathbb{Q}$-vectorial space F generated by $\mathcal{F}$

- with a coproduct defined for all $P \in \mathcal{F}$ by :

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P],$$

- with the direct product of posets.

Theorem (W.R. Schmitt, 1994)

$(F, \Delta, \times)$ *is a Hopf Algebra, called Incidence Hopf algebra.*

We consider the family $\mathcal{P}$ of direct products of partition posets. It is interval closed and stable under direct product.

The associated Incidence Hopf algebra, called *Faa di Bruno Hopf Algebra* is generated by partition posets Π_n. The coproduct in the Incidence Hopf algebra of partition posets is a classical result :

$$\Delta\left(\frac{\Pi_n}{n!}\right) = \sum_{k=1}^{n} \sum_{\substack{(j_1,\dots,j_n)\in\mathbb{N} \\ \sum_{i=1}^n j_i = k, \sum_{i=1}^n i j_i = n}} \binom{k}{j_1, \dots, j_n} \prod_{i=1}^{n} \left(\frac{\Pi_i}{i!}\right)^{j_i} \otimes \frac{\Pi_k}{k!}.$$

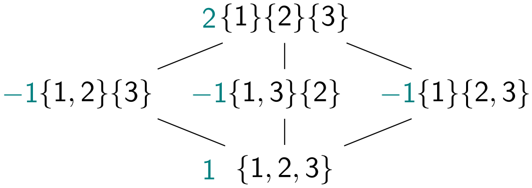
Moebius number

Definition

For any poset P the Moebius function is defined by :

$$\begin{aligned} \mu(x,x) &= 1, && \forall x \in P \\ \mu(x,y) &= -\sum_{x \le z < y} \mu(x,z), && \forall x < y \in P. \end{aligned}$$

If P is bounded, the Moebius number of P is $\mu(P) := \mu(\hat{0},\hat{1})$



Link with Moebius numbers

A character on an incidence Hopf algebra $\mathcal{H}$ is a linear map from $\mathcal{H}$ to $\mathbb{Q}$.
We define the following convolution for all characters ϕ and ψ :

$$\phi * \psi(P) = \sum \phi(P_{(1)})\psi(P_{(2)}),$$

where $\Delta(P) = \sum P_{(1)} \otimes P_{(2)}$, using Sweedler's convention.
The following maps are characters P :

$$\zeta : \Pi_n \mapsto 1,$$

and

$$\mu : \Pi_n \mapsto \mu(\Pi_n).$$

Moreover, if ϵ is the counit of $\mathcal{H}$,

$$\zeta * \mu = \mu * \zeta = \epsilon.$$

Computation of Moebius numbers of partition posets

Computation of the Moebius number of Π_3

$$\mu(1) = 1$$
$$\mu(\Pi_2) = -\mu(1) = -1$$
$$\mu(\Pi_3) = -3\mu(\Pi_2) - \mu(1) = -3 \times (-1) - 1 = 2$$

Let f be a character on P, the Faa Di Bruno Hopf algebra. The correspondence $f \to \sum_{n\geq 1} f(\Pi_{n-1})t^n/n!$ defines an anti-isomorphism onto a subgroup of $\mathbb{Q}[[x]]$. We obtain : $\zeta(t) = \exp(t) - 1$ and therefore

$$\mu(t) = \ln(1+t) = \sum_{n\geq 1}(-1)^{n-1}(n-1)!\frac{t^n}{n!}.$$

Can we do the same type of computation for the hypertree posets ?

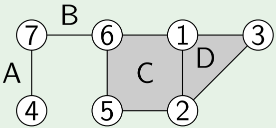
Hypergraphs and hypertrees

Definition (Berge, 1989)

A hypergraph (on a set V) is an ordered pair (V, E) where :

- *V is a finite set (vertices)*
- *E is a collection of subsets of cardinality at least two of elements of V (edges).*

Example of a hypergraph on $[1;7]$



Walk on a hypergraph

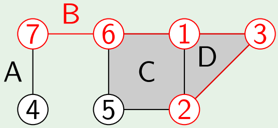
Definition

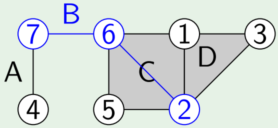
Let $H = (V, E)$ be a hypergraph.
A walk from a vertex d to a vertex f in H is an alternating sequence of vertices and edges beginning by d and ending by f :

$$(d, \dots, e_i, v_i, e_{i+1}, \dots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.

Examples of walks





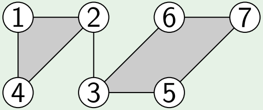
Hypertrees

Definition

A <u>hypertree</u> is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- *there exists a walk from v to w in H with distinct edges e_i, (H is connected),*
- *and this walk is unique, (H has no cycles).*

Example of a hypertree



The hypertree poset

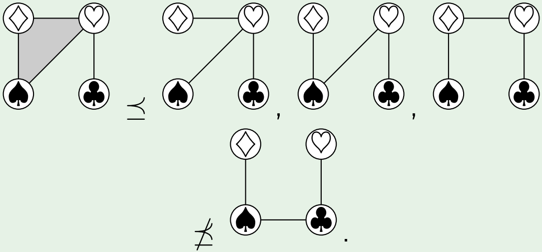
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

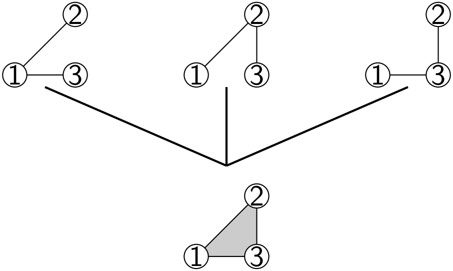
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



Le poset HT_3



- Graded poset by the number of edges [McCullough and Miller 1996],
- Möbius number : $(n-1)^{n-2}$ [McCammond and Meier 2004]
- Action of the symmetric group on the homology [Conjecture Chapoton 2007, B.O. 2013]

Goal :

Construction of an analogue of Incidence Hopf algebra which enables us to compute some characters of posets.

From the Incidence Hopf Algebra to a simpler Bialgebra

Lemma (McCammond, Meier, 2004)

Let τ be a hypertree on n vertices.

(a) *The interval $[\hat{0}, \tau]$ is a direct product of partition posets, with one factor Π_j for each vertex in τ with valency j.*

(b) *The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets, with one factor $\mathrm{HT_j}$ for each edge in τ with size j.*

- Add a maximum element to hypertree posets, products of hypertree posets and products of hypertree and partition posets
- Close by interval and product

$\Rightarrow$ Incidence Hopf algebra $\mathcal{H}$

The Incidence Hopf Algebra of partitions poset is a subbialgebra of this algebra.

From the Incidence Hopf Algebra to a simpler Bialgebra

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- Add a maximum element to hypertree posets, products of hypertree posets and products of hypertree and partition posets
- Close by interval and product

$\Rightarrow$ Incidence Hopf algebra $\mathcal{H}$

Construction of a smaller bialgebra in which computation will be easier.

THE Bialgebra

The family of products of hypertree and partition posets is interval closed and closed by direct product $\rightsquigarrow$ associated algebra $\mathcal{B}$
We endow this algebra with the following coproduct :

$$\Delta(d) = \sum_{x \in d} [\hat{0}_d, x] \otimes [x, \hat{1}_d],$$

for bounded poset $d \in \mathcal{B}$ and

$$\Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\widehat{t}}),$$

for a poset $t \in \mathcal{B}$ with a least element but no greatest one,
where $\widehat{t}$ is the bounded poset obtained from t by adding a greatest element.

Proposition

$\mathcal{B}$ is a bialgebra

Comparison between coproducts

$$h_n = HT_n$$
$$p_n = \Pi_n$$

- Same coproducts on bounded posets.
- In $\mathcal{H}$

$$\Delta(\widehat{t}) = \sum_{x \in \widehat{t}} [\hat{0}, x] \otimes [x, \hat{1}]$$

- In $\mathcal{B}$

$$\Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\widehat{t}})$$

Define :

$$\lambda : \mathcal{B} \to \mathcal{H}, p_k \mapsto p_k, h_j \mapsto \widehat{h_j}.$$

Let α and β be two characters on $\mathcal{H}$ such that $\exists \epsilon_\alpha, \epsilon_\beta \in \mathbb{Q}$ and $\exists \widetilde{\alpha}, \widetilde{\beta} : \mathcal{B} \to \mathbb{Q}$:

$$\alpha(\lambda(p_k)) = \widetilde{\alpha}(p_k),\ \alpha(\lambda(h_j)) = \epsilon_\alpha \widetilde{\alpha}(h_j),$$

and

$$\beta(\lambda(p_k)) = \widetilde{\beta}(p_k),\ \beta(\lambda(h_j)) = \epsilon_\beta \widetilde{\beta}(h_j).$$

Theorem

The convolution of the characters α and β on $\mathcal{H}$ is given by :

$$\alpha * \beta(\lambda(p_k)) = \sum \widetilde{\alpha}(p_k^{(1)})\widetilde{\beta}(p_k^{(2)}),$$

and

$$\alpha * \beta(\lambda(h_j)) = \epsilon_\beta \sum \widetilde{\alpha}(h_j^{(1)})\widetilde{\beta}(h_j^{(2)}) + \epsilon_\alpha \widetilde{\alpha}(h_j),$$

where $\Delta(p_k) = \sum p_k^{(1)} \otimes p_k^{(2)}$ and $\Delta(h_j) = \sum h_j^{(1)} \otimes h_j^{(2)}$ in $\mathcal{B}$.

Computation of the Coproduct in this Bialgebra

$$\Delta(h_n) = \sum_{(\alpha,\pi)\in\mathcal{P}_n} c^n_{\alpha,\pi} p_\alpha \otimes h_\pi,$$

where for all $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\pi = (\pi_2, \pi_3, \dots, \pi_l)$, $p_\alpha = 1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $h_\pi = h_2^{\pi_2} h_3^{\pi_3} \dots h_l^{\pi_l}$.

Lemma (McCammond, Meier, 2004)

Let τ be a hypertree on n vertices.

(a) *The interval $[\hat{0}, \tau]$ is a direct product of partition posets, with one factor Π_j for each vertex in τ with valency j.*

(b) *The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets, with one factor $\mathrm{HT_j}$ for each edge in τ with size j.*

$c^n_{\alpha,\pi} =$ number of hypertrees in h_n with :

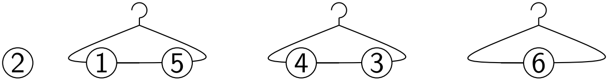
- α_i vertices of valency i, $\forall i \geq 1$
- π_j edges of size j, $\forall j \geq 2$

Criterion for the vanishing of $c^n_{\alpha,\pi}$

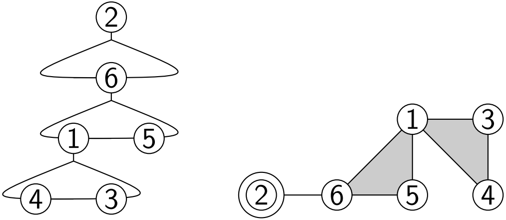
$$c^n_{\alpha,\pi} \neq 0 \Longleftrightarrow \sum_{i=1}^{k} \alpha_i = n, \quad \sum_{j=2}^{l} (j-1)\pi_j = n-1 \text{ and } \sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l} \pi_j - 1.$$

Counting (rooted) hypertrees

A π-hooked partition P, for $\pi = (1, 2)$:



Associated hypertree



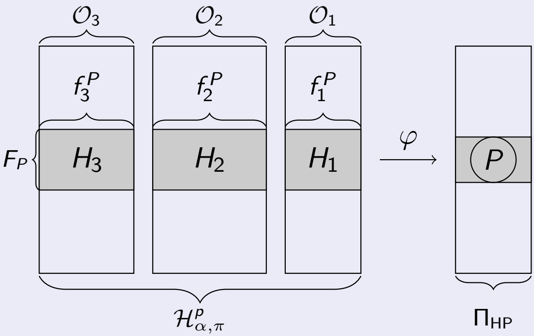
Counting hypertrees

Let us now link hypertrees to hooked partitions.
Let $d^n_{\alpha,\pi}$ be the number of ways to construct a hypertree with valencies α from a given π-hooked partition. Then,

The coefficient $c^n_{\alpha,\pi}$ is linked with $d^n_{\alpha,\pi}$ by :

$$c^n_{\alpha,\pi} = \frac{1}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times d^n_{\alpha,\pi},$$

Démonstration.

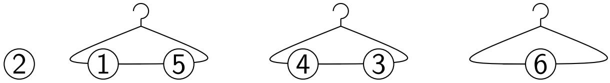


$$c^n_{\alpha,\pi} = \frac{1}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times d^n_{\alpha,\pi},$$

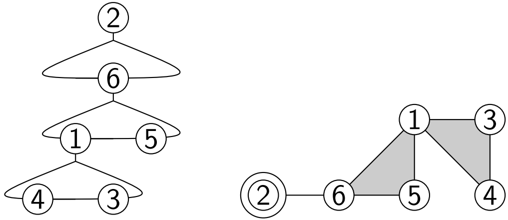
□

Prüfer code

A π-hooked partition P, for $\pi = (1, 2)$:



Code :



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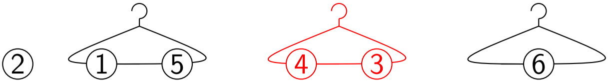

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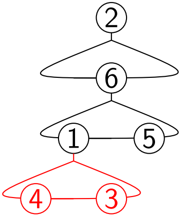

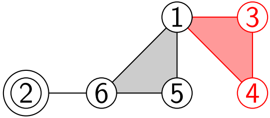

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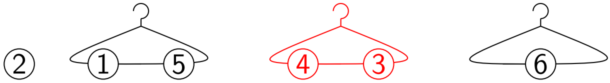
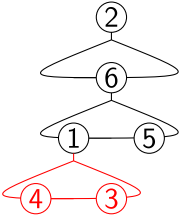
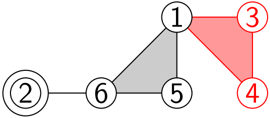


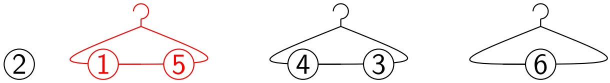
Code : **1**



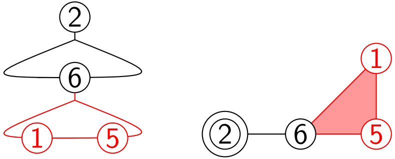


Prüfer code

A π-hooked partition P, for $\pi = (1, 2)$:

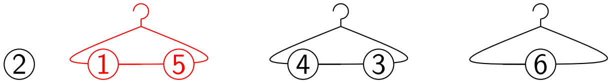


Code : 1,

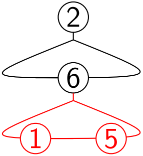


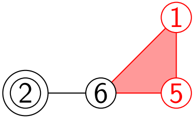
Prüfer code

A π-hooked partition P, for $\pi = (1, 2)$:



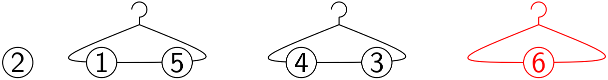
Code : 1, **6**



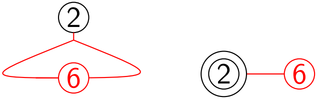


Prüfer code

A π-hooked partition P, for $\pi = (1,2)$:

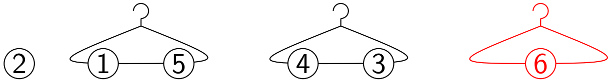


Code : 1, 6,

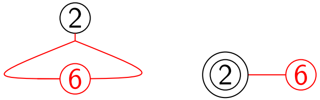


Prüfer code

A π-hooked partition P, for $\pi = (1, 2)$:

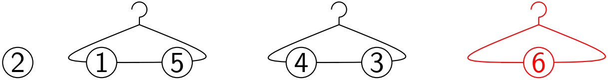


Code : 1, 6, **2**

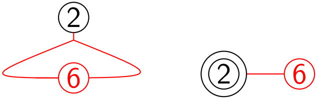


Prüfer code

A π-hooked partition P, for $\pi = (1, 2)$:



Code : 1, 6, **(2)**



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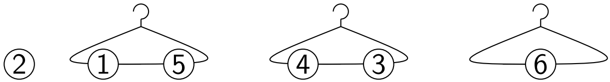
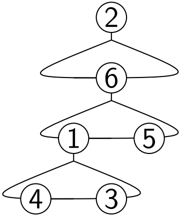
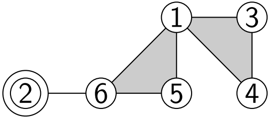


Code : 1, 6





Return of the Prüfer code

constructions of a rooted hypertree of valency set α from P_π

$$\Longleftrightarrow$$

words on $[\![1, n]\!]$, of length $k = \sum_{j\geq 2} \pi_j - 1$, with $\sum_{i \geq 2} \alpha_i$ different letters, where α_i letters appear $i-1$ times, $\forall i \geq 2$

$$\rightsquigarrow \frac{k! \times n!}{\prod_{i\geq 2}(i-1)!^{\alpha_i}\alpha_i!}.$$

Theorem (B.O.)

$$\Delta(h_n) = \frac{1}{n} \times \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{k! \times n!}{\prod_{i\geq 1}(i-1)!^{\alpha_i}\alpha_i!} \prod_{i=2}^{k} p_i^{\alpha_i} \otimes \prod_{j=2}^{l} h_j^{\pi_j}.$$

Application : Computation of Moebius numbers of Hypertree Posets

Theorem (McCammond and Meier 2004)

The Moebius number of the augmented hypertree poset on n vertices is given by :

$$\mu(\widehat{HT_n}) = (-1)^{n-1}(n-1)^{n-2}.$$

The following equality holds :

$$(n-1)^{n-2} = \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{(-1)^{i\alpha_i - 1}}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{k! \times n!}{\prod_{i\geq 1}\alpha_i!},$$

where $\mathcal{P}(n) = (\alpha = (\alpha_1, \dots, \alpha_k), \pi = (\pi_2, \dots, \pi_l))$ satisfying :

$$\sum_{i=1}^{k} \alpha_i = n, \quad \sum_{j=2}^{l}(j-1)\pi_j = n-1, \quad \text{and} \quad \sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l}\pi_j - 1.$$

Thank you very much for your attention !