STRUCTURE THEOREMS FOR DENDRIFORM AND TRIDENDRIFORM ALGEBRAS

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Abstract. We state new Cartier-Milnor-Moore Poincaré-Birkhoff-Witt theorems for dendriform and tridendriform structures. We introduce the terplicial coalgebra structure as an analogue in the tridendriform algebras of the duplicial co-structure for the dendriform case, and prove a rigidity theorem.

Combinatorial objects gain in comprehension when seen as constructible from a few elements, for instance, when seen as a free algebra over a vector space. For a combinatorial object endowed with an associative algebraic operation with respect to an order, one may consider the dendriform and tridendriform algebra associated to it. Considering these finer structures reduces the dimension (rankwise) of the vector space when seen as a free dendriform or tridendriform algebra. But, proving that an object is free over one of these structure is not always easy. Some tools exist, mostly generalised Poincaré-Birkhoff-Witt Cartier-Milnor-Moore theorems or generalised Borel (called *rigidity theorems*). Namely, in the dendriform case with Foissy's work [10] where he considers two different dendriform structures one as an algebra and one as a coalgebra which are not dual from each other; or the work of Loday and Ronco considering an associative coalgebra and a dendriform algebra [17]. One can also consider a different approach and find a duplicial structure (closely related to the dendriform structure) and consider a duplicial-duplicial rigidity theorem due to [16]. But such an analogue would not exist in the tridendriform case. In the tridendriform case, one structure theorem is due to Maria Ronco and the first author when considering an associative structure and a tridendriform algebra.

We investigate different rigidity theorems, namely those arising from the dualisation of the dendriform or tridendriform operations. We provide explicit combinatorial expressions of the intertwining relations which permit rapidly to conclude to the non-freeness of an object and provide a deeper knowledge of it.

We introduce the terplicial algebra, and prove some rigidity theorems with a terplicial coalgebra structure, which plays an analogous role to the duplicial algebra in the dendriform case. The intertwining relations are

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short. Moreover, the free terplicial and the free tridendriform algebra are spanned by trees and the terplicial operation can be seen as a set operation related to the tridendriform one via an order, as defined in [27].

The paper has two parts one dedicated to the dendriform case and the second to the tridendriform case.

It is organised as follows: we recall definitions of the dendriform and duplicial algebra and coalgebras. We introduce the duplicial dendriform bialgebra with their explicit intertwinning relations and state their associated rigidity theorem. To prove the intertwining relations, we give a precise description of the product and coproduct in the free algebra using paths. It introduces a whole new set of operations indexed by paths with their intertwining relations combinatorially explicited. The computation of the relations gives, as a byproduct, the number of elements of some intervals of the Tamari posets considered by Chatel and Pons [7].

The terplicial operad introduced later on is not exactly an analogous of the duplicial operad with regards to the dendriform operad but merely of a skew-symmetric version. We investigate this operad and rigidity theorems associated to it. In the next section, we illustrate on the Solomon-Tits algebra and the Parking functions these above structures and get as a result yet another proof of their freeness as dendriform algebra, see [2, 11, 10, 23, 27].

The second part of the paper is dedicated to the tridendriform case. After recalling the definition of the tridendriform algebra and coalgebra, we introduce the terplicial algebra and coalgebra. We then state the rigidity theorem for terplicial bialgebras, co-terplicial tridendriform bialgebras, and dual tridendriform bialgebras.

The next section is devoted to a precise description of the tridendriform and the terplicial operations in the free algebra and cofree coalgebra through paths in order to prove the intertwining relations as in the dendriform case. It gives a new insight with their dual path indexed coproducts.

1 Dual dendriform bialgebras and duplicial-dendriform bialgebras.

This section is devoted to the dendriform algebra structure. We prove a new rigidity theorem for duplicial-dendriform bialgebras: a vector space with a duplicial coalgebra structure and a dendriform algebra structure which are moreover linked through intertwining relations. We also consider a dendriform bialgebras where the operations and the cooperations are obtained by dualisation. We investigate very precisely the relations between the product and coproduct to have a deeper knowledge of free dendriform algebra as dualising an operation is the most natural thing one can do on a given combinatorial object.

1.1 Confluence law et rigidity theorem We recall from [1] the definitions of a confluence law and a rigidity theorem, which extend the results presented by Loday in [16]. Note that we are considering non-symmetric quadratic operads here, which implies a simplification of definitions. We first recall the usual definitions on operads and algebras, for self-containedness.

Definition 1.1.1. A *(connected non-symmetric) operad* $\mathcal{A} = ((\mathcal{A}(n))_{n\geq 1}, \gamma)$ is

- a graded vector space $\mathcal{A} = \bigoplus_{n\geq 1} \mathcal{A}(n)$, with $\mathcal{A}(1) = \mathbb{K}.\mathbf{1}$ and $\mathcal{A}(n)$ finite dimensional for any $n \geq 1$,
- endowed with linear morphisms, called *(partial) composition maps*, $\circ_i : \mathcal{A}_m \otimes \mathcal{A}_n \to \mathcal{A}_{m+n-1}$ for any $m, n \geq 1$,

such that $\circ_i$ and $\mathbf{1}$ satisfy associativity and unitality, for any elements $x, z \in \mathcal{A}$ and $y \in \mathcal{A}_l$:

$$\begin{aligned} (x \circ_j z) \circ_i y &= (x \circ_i y) \circ_{j+l-1} z && \text{if } i < j \\ x \circ_i (y \circ_j z) &= (x \circ_i y) \circ_{i+j-1} z && \text{if } 1 \leq j \leq l \\ \mathbf{1} \circ_1 x &= x = x \circ_i \mathbf{1}, && \text{for any } 1 \leq i \leq m \end{aligned}$$

Let us first recall the definition of an algebra, a cooperad and a coalgebra over an operad.

Definition 1.1.2. An *algebra* over an operad $\mathcal{A}$ is a vector space A equipped with a linear morphism $m_A^n : \mathcal{A}(n) \otimes A^{\otimes n} \to A$. We denote the *free algebra* over an operad $\mathcal{A}$ whose vector space of generators is V by

$$\mathcal{A}(V) = \bigoplus_{n\geq 1} \mathcal{A}(n) \otimes_{\mathfrak{S}_n} \left(\underbrace{V \otimes \ldots \otimes V}_{n} \right) = \bigoplus_{n\geq 1} \mathcal{A}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}. \tag{1}$$

Definition 1.1.3. A *coalgebra* over an operad $\mathcal{C}$ is a vector space C equipped with a linear morphism $\gamma_C^n : \mathcal{C}(n) \otimes C \to C^{\otimes n}$. We denote the *free (conilpotent) coalgebra* over an operad $\mathcal{C}$ (or equivalently its associated dual cooperad $\mathcal{C}^*$) whose vector space of primitives is V by

$$\mathcal{C}^c(V) = \bigoplus_{n\geq 1} \mathcal{C}^*(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}. \tag{2}$$

Note that the coproducts considered in this article are reduced, i.e. obtained by removing from the coproduct terms in which the unit of the algebra appears, if it exists. We need moreover the following notion:

Definition 1.1.4. The cofiltration $\mathcal{F}_n\mathcal{H}$ can be defined on any $\mathcal{C}^c$-coalgebra $\mathcal{H}$:

$$\mathcal{F}_n\mathcal{H} = \{x \in \mathcal{H} | \forall p > n, \forall \delta \in \mathcal{C}(p), \delta(x) = 0\}.$$

The vector space $\mathcal{F}_1\mathcal{H}$ is the vector space of *primitive elements*. Moreover, we denote by $\iota_{\mathcal{H}} : \mathcal{F}_1\mathcal{H} \to \mathcal{H}$ the canonical inclusion.

A $\mathcal{C}^c$-coalgebra $\mathcal{H}$ is said to be *conilpotent* if $\mathcal{H} = \cup_{n\geq 1}\mathcal{F}_n\mathcal{H}$.

Definition 1.1.5. A *confluence law* α between operads $\mathcal{A}$ and $\mathcal{C}$ is a family of maps

$$\alpha_{m,n} : \mathcal{C}(m) \otimes \mathcal{A}(n) \to \bigoplus_{n_1+\ldots+n_m=n} \mathcal{A}(n_1) \otimes \ldots \otimes \mathcal{A}(n_m), \tag{3}$$

such that $\alpha_{m,n}$ is compatible with the structure of operad of $\mathcal{C}$.

Confluence laws are a generalisation of mixed distributive laws as defined by Fox and Markl in [12].

Example 1.1.6. It has been proven in [1] that a family satisfying a rigidity theorem (as defined below) is the family of preLie (product $\curvearrowleft$) copreLie (coproduct Δ) bialgebras with confluence laws given on $T \in \mathrm{PreLie}(n)$ and $S \in \mathrm{PreLie}(k)$ by:

$$\Delta (T \curvearrowleft S) = n \times T \otimes S + (T \curvearrowleft S_1) \otimes S_2 + (T_1 \curvearrowleft S) \otimes T_2 + T_1 \otimes (T_2 \curvearrowleft S),$$

where $\Delta(T) = T_1 \otimes T_2$ and $\Delta(S) = S_1 \otimes S_2$.

Let us now define $\mathcal{C}^c -_\alpha \mathcal{A}$-bialgebras.

Definition 1.1.7. A $\mathcal{C}^c -_\alpha \mathcal{A}$-bialgebra H is a $\mathbb{K}$-vector space H endowed with a structure of $\mathcal{A}$ -algebra, a structure of $\mathcal{C}^c$-coalgebra, satisfying a confluence law.

Remark 1.1.8. Note that given a type of algebra and a type of coalgebra, there can be several choices of confluence law, none of them canonical.

Theorem 1.1.9 ([1], Rigidity theorem)**.** *Let $\mathbb{K}$ be a field of characteristic 0 and let us consider two connected algebraic operads $\mathcal{A}$ and $\mathcal{C}$, such that $\mathcal{A}(n)$ and $\mathcal{C}(n)$ are finite dimensional vector spaces. To any family of isomorphisms $\varphi_n : \mathcal{A}(n) \to \mathcal{C}^*(n)$ can be associated a confluence law (α) such that any conilpotent $\mathcal{C}^c -_\alpha \mathcal{A}$-bialgebras is free and cofree over the vector space of its primitive elements*

$$\mathcal{A}(\mathrm{Prim}\,\mathcal{H}) \cong \mathcal{H} \cong \mathcal{C}^c(\mathrm{Prim}\,\mathcal{H}).$$

1.2 Definitions: dendriform and duplicial (co)algebras. We recall the definitions of dendriform algebras, dendriform coalgebras and describe the free dendriform algebra on the vector space spanned by the planar binary rooted trees PBT, and the conilpotent cofree codendriform coalgebra. We recall the definitions of duplicial algebras, coalgebras and describes their free algebra and cofree conilpotent coalgebra on PBT.

Definition 1.2.1. A *dendriform algebra* (see [15]) structure on a vector space A is a pair of binary products $\prec: A \otimes A \to A$ and $\succ: A \otimes A \to A$, satisfying that:

$$\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c),\\
(a \succ b) \prec c &= a \succ (b \prec c),\\
(a \prec b + a \succ b) \succ c &= a \succ (b \succ c).
\end{aligned}$$

Example 1.2.2. For any vector space V, the free dendriform algebra over V, denoted by $PBT(V)$, was defined in [17]. Its underlying vector space is spanned by labelled planar binary trees, where the vertices of planar rooted trees are labelled with the elements of a basis of V.

For the sake of clarity, we recall that a *planar rooted tree* is a tree with a distinguished vertex called the root, such that all edges are oriented away from the root. A tree will be drawn with the root at the bottom. For an oriented edge from a vertex a to a vertex b, the vertex a will be called the source and b the target. The vertex a is then the parent of b and b a child of a. A *leaf* is a vertex which is note a source and an edge is said to be *inner* if its target is not a leaf. Vertices will be either leaves of sources of more than two edges (exactly two in the case of binary trees). We denote the vertices of a rooted tree t by $V(t)$, its root by $r(t)$ and its edges by $E(t)$.

Definition 1.2.3. Given an integer $n \geq 2$, the grafting of trees $t_1, \ldots, t_n$ on a root is the tree whose vertex set is $\bigcup_{i=1}^{n} V(t_i) \cup \{\bullet\}$, whose edge set is $\bigcup_{i=1}^{n} E(t_i) \cup \bigcup_{i=1}^{n} \{(\bullet, r(t_i))\}$ and whose root is $\bullet$. We denote it by $\vee(t_1, \ldots, t_n)$.

Any planar rooted tree T is the grafting of two trees of smaller degree, t_l and t_r ; that is $T = \vee(t_l, t_r)$. The products $\prec, \succ$ are defined recursively on any trees T and S by:

$$T \succ \emptyset = \emptyset \prec T = T \ , \quad \emptyset \succ T = T \prec \emptyset = \emptyset,$$

$$T \prec S = \vee(t_l, (t_r \prec S + t_r \succ S)) \ , \tag{4}$$

$$T \succ S = \vee((T \prec s_l + T \succ s_l)\, , s_r). \tag{5}$$

A *codendriform coalgebra* is a vector space C with two cooperations $\Delta_{\prec}$ and $\Delta_{\succ}$ satisfying equations obtained by dualizing the one in Definition 1.2.1.

Example 1.2.4. As $PBT(\mathbb{K})$ is a dendriform algebra, its graded dual $PBT(\mathbb{K})^*$ is a codendriform coalgebra. The isomorphism between $PBT(\mathbb{K})$ and $PBT(\mathbb{K})^*$ given by the basis of planar binary trees on $PBT(\mathbb{K})$ induces a codendriform coalgebra structure on $PBT(\mathbb{K})$. For any tree T, the codendriform cooperation is then given by : $\Delta_{\succ}(T) = \sum T_{(\succ 1)} \otimes T_{(\succ 2)}$ (resp. $\Delta_{\prec} = \sum T_{(\prec 1)} \otimes T_{(\prec 2)}$), where the sum runs over all pairs of planar binary trees $(T_{(\succ 1)}, T_{(\succ 2)})$ satisfying $T^*(T_{(\succ 1)} \succ T_{(\succ 2)}) \neq 0$ (resp. $T^*(T_{(\prec 1)} \prec T_{(\prec 2)}) \neq 0$). The definition of the operations gives a constructive way to define the cooperations on a tree $T = \vee(t_l, t_r)$:

$$\Delta_{\succ}(T) = t_l \otimes (\vee(\emptyset, t_r)) + \sum (t_l)_{(*1)} \otimes \vee\left((t_l)_{(*2)}, t_r\right) \tag{6}$$

$$\Delta_{\prec}(T) = \vee\,(t_l, \emptyset) \otimes t_r + \sum \vee\left(t_l, (t_r)_{(*1)}\right) \otimes (t_r)_{(*2)} \tag{7}$$

where $\Delta_* = \Delta_{\prec} + \Delta_{\succ}$, $\Delta_*(t) = \sum t_{(*1)} \otimes t_{(*1)}$, and $\Delta_{\succ}(\vee(\emptyset, t_r)) = \Delta_{\prec}(\vee(t_l, \emptyset)) = 0$. This can be proven by direct inspection. The above coalgebra structure on the planar binary trees is the free conilpotent codendriform coalgebra. The proof is straightforward by dualisation.

Definition 1.2.5. A *duplicial algebra* structure on A is a pair of binary products $\triangleright : A \otimes A \to A$ and $\triangleleft : A \otimes A \to A$, satisfying that:

$$\begin{gathered}\triangleright \text{ and } \triangleleft \text{ are associative,}\\ (x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z), \text{ for any x,y,z in } A.\end{gathered}$$

Example 1.2.6. The free duplicial structure on the planar binary rooted trees is given, for any trees T and S, by:

$$\begin{aligned}&T \triangleright \emptyset = \emptyset \triangleleft T = T\ , \quad \emptyset \triangleright T = T \triangleleft \emptyset = \emptyset\\ &T \triangleright S = \vee((T \triangleright s_l)\, , s_r) \text{ and}\\ &T \triangleleft S = \vee((t_l, (t_r \triangleleft S))\end{aligned}$$

A *coduplicial coalgebra* is a vector space C with two cooperations $\Delta_\triangleleft$ and $\Delta_\triangleright$ satisfying equations obtained by dualizing the one in Definition 1.2.5. Note that coduplicial coalgebras appear in [14] under the name of L-coalgebras.

Example 1.2.7. The free conilpotent coduplicial coalgebra is isomorphic to the vector space generated by planar binary rooted trees, endowed with the coproducts:

$$\begin{aligned}&\Delta_\triangleright(T) = t_l \otimes \vee(\emptyset, t_r) + (t_l)_{(\triangleright 1)} \otimes \vee\Big((t_l)_{(\triangleright 2)}, t_r\Big)\\ &\Delta_\triangleleft(T) = \vee(t_l, \emptyset) \otimes t_r + \vee\Big(t_l, (t_r)_{(\triangleleft 1)}\Big) \otimes (t_r)_{(\triangleleft 2)},\\ &\text{with } \Delta_\triangleleft(\vee(t_l, \emptyset)) = \Delta_\triangleright(\vee(\emptyset, t_r)) = 0.\end{aligned}$$

1.3 Duplicial-dendriform bialgebras. Let $\mathcal{H}$ be a vector space with a dendriform algebra structure $(\mathcal{H}, \prec, \succ)$ and a coduplicial coalgebra structure $(\mathcal{H}, \Delta_\triangleright, \Delta_\triangleleft)$. Let us now determine the confluence law associated with the previously introduced products and coproducts on planar binary trees. Note that in [11], a confluence law is introduced for duplicial codendriform bialgebras. However, it differs from the dual of the one presented here as there is no term $x \otimes y$ in $\Delta_\prec(x \triangleleft y)$.

Definition 1.3.1. Let $\mathcal{H}$ be a vector space with a dendriform algebra structure $(\mathcal{H}, \prec, \succ)$ and a coduplicial coalgebra structure $(\mathcal{H}, \Delta_\triangleright, \Delta_\triangleleft)$. If it satisfies moreover $\forall x, y \in \mathcal{H}$ that:

$$\Delta_\triangleright(x \succ y) = x \otimes y + \Big(x * y_{\triangleright(1)}\Big) \otimes y_{(\triangleright 2)} + x_{(\triangleright 1)} \otimes \Big(x_{(\triangleright 2)} \succ y\Big)\,, \tag{8}$$

$$\Delta_\triangleright(x \prec y) = x_{(\triangleright 1)} \otimes \Big(x_{(\triangleright 2)} \prec y\Big)\,, \tag{9}$$

$$\Delta_\triangleleft(x \succ y) = \Big(x \succ y_{(\triangleleft 1)}\Big) \otimes y_{(\triangleleft 2)}, \tag{10}$$

$$\Delta_\triangleleft(x \prec y) = x \otimes y + \Big(x \prec y_{(\triangleleft 1)}\Big) \otimes y_{(\triangleleft 2)} + x_{(\triangleleft 1)} \otimes \Big(x_{(\triangleleft 2)} * y\Big)\,, \tag{11}$$

where $* = \succ + \prec$, then $\mathcal{H}$ is said to be a *coduplicial-dendriform bialgebra.*

Proposition 1.3.2. *The relations introduced in the previous definition are a confluence law.*

Proof. The only property that has to be checked is the compatibility with the operadic structure. This comes from the fact that these relations are computed on planar binary trees. Equations (9) and (10) comes from the fact that the product is done on one of the two subtree of the root and the coproduct on the other. The two other equations are obtained by noting that in $x \prec y$ (resp. $y \succ x$), the path from the root to the rightmost (resp. leftmost) leaf is first made of edges of x, and then of edges of y. The first term of the right side is obtained when the coproduct does a pruning of the tree at the leaf of x, the second, when it cuts an edge in y and the last when it cuts an edge in x. $\square$

Note that the confluence law obtained is in fact mixed distributive laws as studied in [16]. We can then apply the rigidity theorem in [16] to get:

Proposition 1.3.3. *Any conilpotent coduplicial-dendriform bialgebra is free and cofree over the vector space of its primitive elements.*

1.4 Conilpotent dual dendriform bialgebras are free and cofree.

In [10], Foissy considers dendriform bialgebras: bialgebras with a dendriform structure for the algebraic and the coalgebraic structure with a confluence law linking them both. But, the confluence law is not the one obtained when considering dendriform products and their dual coproducts. Indeed, it is obtained by considering dendriform products and the dualisations of other dendriform products (see [1] for more details). It seems natural to consider the case where the coalgebraic structure is the dual structure of the algebraic structure. In this section we focus on constructing the confluence law that links both structures.

Let $\mathcal{H}$ be a vector space with a dendriform algebra structure $(\mathcal{H}, \prec, \succ)$ and a codendriform coalgebra structure $(\mathcal{H}, \Delta_\prec, \Delta_\succ)$.

We will introduce some notations:

$$\begin{aligned}
&\Delta_\succ^{k+1} = (id^{\otimes k} \otimes \Delta_\succ) \circ \cdots \circ (id \otimes \Delta_\succ) \circ \Delta_\succ \\
&\succ^{k+1} = \succ \circ (id \otimes \succ) \circ \cdots \circ (id^{\otimes k} \otimes \succ) \\
&\alpha_\succ = \Delta_\succ + \sum_{k=1} (-1)^k (id \otimes \succ^k) \circ \Delta_\succ^{k+1} \\
&\alpha_\prec = \Delta_\prec + \sum_{k=1} (-1)^k (\prec^k \otimes id) \circ \Delta_\prec^{k+1}
\end{aligned}$$

where $\alpha_\succ, \alpha_\prec$ are called *alternating convolution operations*, from which we deduce the idempotents $\prec \circ \alpha^\prec$ and $\succ \circ \alpha^\succ$ which satisfy good properties (cf. [16, proposition 2.3.5]).

Note that for any tree $T = \vee(t_l, t_r)$, $\alpha_\succ(T)$ is exactly $t_l \otimes \vee(\emptyset, t_r)$. This is proven by induction while using the definitions (6), (7). Analogously, $\alpha_\prec(T)$ is $\vee(t_l, \emptyset) \otimes t_r$.

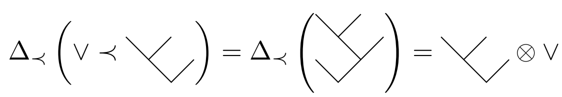
Definition 1.4.1. Let $\mathcal{H}$ be a vector space with a dendriform algebra structure $(\mathcal{H}, \prec, \succ)$ and a codendriform coalgebra structure $(\mathcal{H}, \Delta_\prec, \Delta_\succ)$. If it satisfies moreover that for all $x, y \in \mathcal{H}$:

$$
\begin{aligned}
&\Delta_{\succ}(x \succ y) = \alpha_{\succ}(x \succ y) + \Delta_{*}(x * \alpha_{\succ}(y)_{(\succ 1)})_{(*1)} \otimes \Delta_{*}(x * \alpha_{\succ}(y)_{(\succ 1)})_{(*2)} \succ \alpha_{\succ}(y)_{(\succ 2)} \ ,\\
&\Delta_{\succ}(x \prec y) = \Delta_{\succ}(x)_{(\succ 1)} \otimes \Delta_{\succ}(x)_{(\succ 2)} \prec y \ ,\\
&\Delta_{\prec}(x \succ y) = x \succ \Delta_{\prec}(y)_{(\prec 1)} \otimes \Delta_{\prec}(y)_{(\prec 2)} \ ,\\
&\Delta_{\prec}(x \prec y) = \alpha_{\prec}(x \prec y) + \alpha_{\prec}(x)_{(\prec 1)} \prec \Delta_{*}(\alpha_{\prec}(x)_{(\prec 2)} * y)_{(*1)} \otimes \Delta_{*}(\alpha_{\prec}(x)_{(\prec 2)} * y)_{(*2)} \ .
\end{aligned}
$$

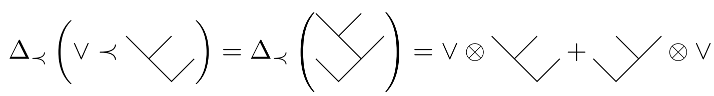
then $\mathcal{H}$ is said to be a dual dendriform bialgebra.

Note that this confluence law is not a finite sum of composition of tensors of operations and cooperations, but applied on an element x it is polynomial.

Remark 1.4.2. One quick way to check that these relations differ from Foissy's one is that in Foissy's relations there is no term $x \otimes y$ in $\Delta_{\prec}(x \prec y)$, but a term $y \otimes x$ instead. This means that on PBT for instance, we have in Foissy's work:

(12) 

and here,

(13) 

Proposition 1.4.3. *The relations introduced in the previous definition are a confluence law.*

Proof. The only property that has to be checked is the compatibility with the operadic structure. This comes from the fact that these relations are computed on planar binary trees. $PBT(V)$ endowed with the above codendriform and dendriform structure is a dual dendriform bialgebra. Indeed, the definition of coproducts of trees (6), (7) applied on the operations $\succ$ and $\prec$ of two trees $T = \vee(t_l, t_r), S = \vee(s_l, s_r)$ give the confluence laws when using the above note:

$$\Delta_{\succ}(T \succ S) = (T * s_l) \otimes \vee(\emptyset, s_r) + (T * s_l)_{(*1)} \otimes \left(\vee\left(T * s_l\right)_{(*2)}, s_r\right) \ , \tag{14}$$

$$\Delta_{\prec}(T \prec S) = \vee(t_l, \emptyset) \otimes (t_r * S) + \vee\left(t_l, (t_r * S)_{(*1)}\right) \otimes (t_r * S)_{(*2)} \ . \tag{15}$$

The relations $\Delta_{\succ}(T \prec S) = T \succ \Delta_{\prec}(S)_{(\prec 1)} \otimes \Delta_{\prec}(S)_{(\prec 2)}$, and $\Delta_{\prec}(T \succ S) = \Delta_{\succ}(T)_{(\succ 1)} \otimes \Delta_{\succ}(T)_{(\succ 2)} \prec S$ come from the fact that the product is done on one of the two subtree of the root and the coproduct on the other. $\square$

Note that this type of confluence law does not fit into the scope of Loday's work in [16]. We then apply rigidity theorem from [1]:

Proposition 1.4.4. *Any conilpotent dual dendriform bialgebra is free and cofree over its primitive.*

2 Combinatorial description of the products and coproducts and the confluence law on PBT.

In the following section, we investigate combinatorically the confluence laws on bialgebras of the planar rooted trees. These descriptions are based on some shuffle paths and cutting paths in trees and are useful to give an explicit confluence law in terms of trees and coefficients, binomials, only. We then relate these coefficients to the work of Chatel and Pons on the Tamari Lattice.

2.1 Cutting paths and shuffle paths. The codendriform cooperations can be described in terms of cutting paths, and the dendriform operations in terms of shuffle paths. We describe more precisely these paths.

2.1.1 Cutting paths: Consider $T = \vee(t_l, t_r)$ a tree and q a path in the tree from the root to a leaf, denote e^i its edges. Note that this path only depends on the choice of a leaf. The orientation of the tree defines for every vertex the notion of a right edge and a left edge. This path is referred to as a *cutting path*. We now construct a coproduct indexed by this cutting path as follows.

If the path q is the leftmost path of T, denoted by l_T, we will define

$$\Delta_{l_T}(T) = \emptyset \otimes T$$

and the path is the rightmost path of T, denoted by r_T, we will define

$$\Delta_{r_T}(T) = T \otimes \emptyset\ .$$

For any other path, denoted $(e_1, \ldots, e_m)$, define inductively the coproduct as follows:

(16)

$$\Delta_{(e_1,\ldots,e_m)}T = \begin{cases} \vee(t_l, \Delta_{(e_2,\ldots,e_m)}(T^{e_1})_{(1)}) \otimes \Delta_{(e_2,\ldots,e_m)}(T^{e_1})_{(2)} & e_1 \text{ is a right edge,} \\ \Delta_{(e_2,\ldots,e_m)}(T^{e_1})_{(1)} \otimes \vee(\Delta_{(e_2\ldots,e_m)}(T^{e_1})_{(2)}, t_r) & e_1 \text{ is a left edge,} \end{cases}$$

using Sweedler's notation for the coproduct and where m is the total number of edges of q and T^{e_1} is the right (resp. left) subtree of T if e_1 is a right edge (resp. left).

Note that this definition is the dual of the definition of the dendriform operations defined in [18]. Considering a cutting path q, the coproduct Δ_q of a tree can be understood as follows: duplicate the cutting path and consider the right-handside of the cutting path in the tree with the cutting path tensored with the cutting path and the left-handside of the cutting path.

For a tree T we will denote $Q(T)$ the set of all paths from root to a non-extremal leaf of T, by Q_l (resp. Q_r) the subset of paths with their first edge being a left (resp. right) edge. Then, the co-operations defined in example 1.2.4 verify:

$$\Delta_*(T) = \sum_{q\in Q(T)} \Delta_q(T)\ , \Delta_\succ(T) = \sum_{q\in Q_l(T)} \Delta_q(T)\ , \Delta_\prec(T) = \sum_{q\in Q_r(T)} \Delta_q(T)\ .$$

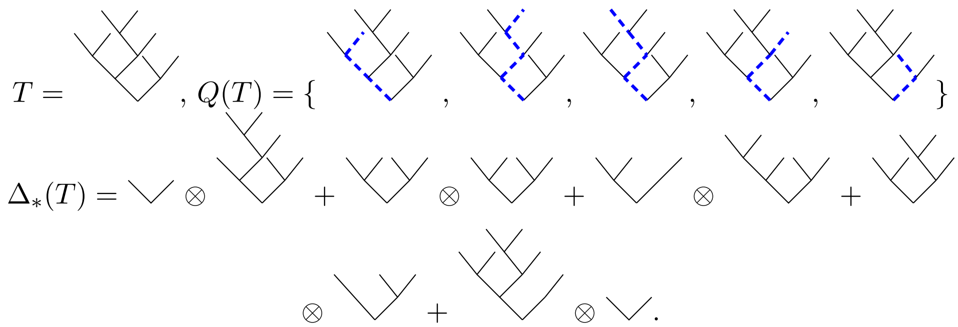


FIGURE 1. Δ_* in terms of cutting paths.

Indeed, it is immediate when considering the construction of the coproduct (7) and (6).

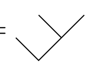
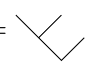
An example is illustrated in figure 1.

2.1.2 Shuffle paths: We refer for example to [16, chapter 5] for the definition of the operations *under* $\backslash$ and *over* $/$ on planar binary trees. Endowed with these operations the PBT is free as a duplicial algebra see [16, proposition 5.1.3]. Any tree T can be uniquely written as $T = t^1/t^2/\dots/t^m$ (resp. $T = t_n\backslash\dots\backslash t_2\backslash t_1$) where t_i, $1 \leq i \leq m$, cannot be written as S/U (resp. t_j, $1 \leq j \leq n$, cannot be written as $S\backslash U$), for any $S, U \in$ PBT.

Consider two trees $T = t^1/\cdots/t^m$ and $S = s_{m+n}\backslash\cdots\backslash s_n$, and a shuffle of the indices $p \in Sh(m,n)$ written as a list of its images $(p(1)\dots,p(m+n))$. We will define a product indexed by a shuffle p, denoted by $T{*_p}S$, inductively as follows:

(17)
$$T *_p S = \begin{cases} t^1/((t^2/\dots/t^m) *_{(p(2),\dots,p(m+n))} S) & \text{if } p(1) = 1 \\ (T *_{(p(2),\dots,p(m+n))} (s_{m+n}\backslash\dots\backslash s_{m+2}))\backslash s_{m+1} & \text{if } p(1) = m+1 \end{cases},$$

with the base case being :

(18) $\vee *_{(1,2)} \vee =$  , $\vee *_{(2,1)} \vee =$  .

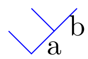
For an example of a shuffle product between two trees, see example 2.1.1.

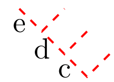
For $p \in Sh(m,n)$ one can associate a path $\tilde{p}$ in the product $T *_p S$ which will be referred to as a *shuffle path*. In the following sequel we will consider the associated paths instead of the shuffles, as we will consider intersections of cutting paths and shuffle paths to describe the confluence laws.

For two trees T, S, with $T = t^1/\cdots/t^m$ and $S = s_{m+n}\backslash\cdots\backslash s_n$, we will denote $Sh(n,m)$ the set of $n+m$ shuffles and by $Sh_<(n,m)$ (resp. $Sh_>(n,m)$) the shuffle $p = (p(1),\dots,p(n+m))$ verifying $p(1) = 1$ (resp. $p(1) = m+1$). Then, the dendriform operations verify, see [18],

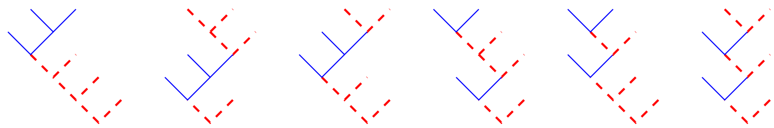
$$T{*}S = \sum_{p\in Sh(m,n)} T{*_p}S\ ,\ T \succ S = \sum_{p\in Sh_>(m,n)} T{*_p}S\ , T \prec S = \sum_{p\in Sh_<(m,n)} T{*_p}S\ .$$

The proof is immediate when considering the constructive definition of the product (4) and (5).

Example 2.1.1. For $T =$ 

and $S =$ 

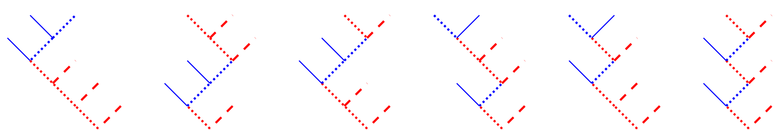
, the set of right shuffle paths is $P_{>}(T,S) = \{\{c,d,e,a,b\},\{c,a,b,d,e\},\{c,d,a,b,e\},\{c,a,d,e,b\},\{c,d,a,e,b\},$ $\{c,a,d,b,e\}\}$ then $T \succ_p S$, with $p \in P_{>}(T,S)$, are given respectively by:



2.2 Confluence law with non-constant coefficients on PBT. Understanding on PBT the confluence laws in a more combinatorial manner, with confluence laws with non-constant coefficients but depending on the cofiltration of the coalgebra is an efficient tool to determine either the structure "should" have a free structure on a given combinatorial object or to show rapidly that it is not free.

Consider the free cofree conilpotent dual dendriform bialgebra on PBT with combinatorially defined products and coproducts. Understanding in a more combinatorial way the confluence law asks to compute the coefficients arising in front of the different elements appearing in it.

Example 2.2.1. In the example 2.1.1, one gets that the number of elements of $T \otimes S$ in the coproduct of $\Delta_{\succ}(T \succ S)$ is 6 which correspond to the cutting paths drawn with dots:



Note that the cutting paths and the shuffle paths coincide.

From the above example, it becomes clear that elements in the coproduct $\Delta_{\succ}(T \succ S)$ will appear multiple times: it depends on the intersection of the cutting path and the shuffle path.

Therefore, the first step is to understand the confluence law between the product associated to a shuffle path and the coproduct associated to a cutting path.

2.2.1 Confluence laws between shuffle paths and cutting paths We will denote the set of edges of a path p as $E(p)$.

Let T, S be two trees. Consider p a shuffle path of $P(T,S)$, and q cutting path of $Q(T *_p S)$. The intersection between the two paths $p \cap q$ is a (possibly empty) path in $T *_p S$ with edges $E(p) \cap E(q)$. Consider also the remaining of the cutting path, denoted q^c: it is the path in $T *_p S$ composed of the edges $E(q) \setminus (E(p) \cap E(q))$.

Lemma 2.2.2. *This path, q^c, is empty or a path with edges strictly in T or in S according to the reached leaf.*

We will denote that path q_T (resp. q_S) if its edges are in T (resp. in S).

Proof. We will denote by r_T the rightmost path of the tree T, and by l_S the leftmost path of the tree S.

The lemma is proven by induction on the number of leaves using the combinatorial definitions of the product (17) and coproduct (16). Indeed, in low dimensions it is clear. Suppose the property true for trees with the sum of their number of leaves inferior or equal to n. Consider two trees $T = \vee(t_l, t_r)$ and $S = \vee(s_l, s_r)$ such that the total number of their leaves is $n + 1$, with a shuffle path p denoted as a sequence of edges $(p^i)_{1\leq i\leq n}$ and a cutting path q denoted as a sequence of edges $(q^i)_{1\leq i\leq m}$.

The following cases can occur: if $q^1 = p^1$ then consider the paths $(p^2, \ldots, p^n)$ and $(q^2, \ldots, q^m)$ in the trees $T^1 = t_r$, $S^1 = S$ if p^1 is the first edge of r_T or $T^1 = T$, $S^1 = s_l$ if p^1 is the first edge of l_S and conclude by induction.

If $q^1 \neq p^1$, suppose moreover that p^1 is an edge of r_T (the symmetric being p^1 is an edge of l_S). Then by construction q^1 is the left edge of the root of T. Moreover $E(q) \subset E(T) \setminus E(r_T)$ as $\Delta_q(T *_p S) = \Delta_{(q^2,\ldots,q^m)}(t_l)_1 \otimes \vee(\Delta_{(q^2,\ldots,q^m)}(t_l)_2, t_r *_{(p^2,\ldots,p^n)} S)$. □

As mentioned above the coproduct of a tree along a cutting path can be understood as thickening the cutting path and cutting it in two to give each side of the tensor. We therefore need to take more notations to describe the confluence law :

When q^c is a path of T we will denote by $[q]_T$ the path in T defined by the sequence of edges $E(q) \cap E(T)$ and $[q]_S$ is defined analougously.

Denote p^c the remaining of the shuffle path in $T *_p S$, i.e. the path defined by the sequence of edges of $E(p) \setminus (E(p) \cap E(q))$. The intersection path $p \cap q$ has edges in S and in T. We will denote by p_S the sequence of edges $(E(p) \cap E(q) \cap E(S)) \cup E(p^c)$ which is the trace of the shuffle path in S, and analogously defined p_T the sequence of edges $(E(p) \cap E(q) \cap E(T)) \cup E(p^c)$.

Lemma 2.2.3.
$$\Delta_q(T *_p S) = \begin{cases} T \otimes S & \textit{if } p = q \\ \Delta_{[q]_T}(T)_1 \otimes \Delta_{[q]_T}(T)_2 *_{p_S} S & \textit{else if } q^c = q_T \\ T *_{p_T} \Delta_{[q]_S}(S)_1 \otimes \Delta_{[q]_S}(S)_2 & \textit{else if } q^c = q_S \end{cases}$$

Proof. A cutting path is determined by the leaf to which it leads, and by definition of $T *_p S$, this leaf is a former leaf of T or of S. Therefore, all the cases are considered. The lemma is proven by induction on the number of leaves, and consider the two cases whether or not the first edge of the shuffle path and the cutting path coincide and then by applying (16) to (17). □

2.2.2 Confluence laws on co-dendriform dendriform bialgebras: Consider r_T the rightmost path of T, denote R_T its number of edges and $(e_i^T)_i$ its edges $1 \leq i \leq R_T$. Respectively denote $L_S = \# l_S$ to be the number of edges of the leftmost path l_S of a tree S, and denote $(e_i^S)_i$ its edges.

Corollary 2.2.4. *The number of terms of $T \otimes S$ in $\Delta_{\succ}(T \succ S)$ (resp. in $\Delta_{\prec}(T \prec S)$) is $\binom{R_T+L_S-1}{R_T}$ (resp. $\binom{R_T+L_S-1}{L_S}$).*

Proof. From the above lemma 2.2.3, the element $T \otimes S$ will appear when the shuffle path and the cutting path coincide. It suffices to compute the cardinal of $P_{<}(T,S)$ which is the collection of shuffles of two sequences of cardinal R_T and $L_S - 1$, i.e. $\binom{R_T+L_S-1}{R_T}$. $\square$

Proposition 2.2.5. *For two planar binary trees T and S, the edges of the rightmost path of T will be denoted $(e_i^T)_i$ and the edges of the leftmost path of S will be denote $(e_i^S)_i$.*

Denote by q_i^T (respectively q_i^S) the cutting path such that i is the maximal integer such that the first i edges are the first i edges of r_T (respectively of l_S), with $0 \leq i \leq R_T$, i.e. $q_i^T = (e_1^T, \ldots, e_i^T, q_{i+1}, \ldots, q_{|q|})$ with q_{i+1} an edge of S , denote p_j^T (respectively p_j^S) the shuffle path such that j is the maximal integer such that the first j edges are edges of r_T (resp. of l_S) with $0 \leq j \leq L_S$.

The confluence law on PBT is given by:

$$\begin{aligned}
\Delta_{\succ}(T \succ S) &= \binom{R_T + L_S - 1}{R_T} T \otimes S \\
&\quad + \sum_{\substack{q \,\in\, Q(T) \\ q \,=\, q_i^T}} \; \sum_{\substack{p \,\in\, P_{<}(T^{e_i^T}, S) \\ p \,=\, p_j^S}} \binom{i+j-1}{i} \Delta_q(T)_1 \otimes \Delta_q(T)_2 *_p S \\
&\quad + \sum_{\substack{q \,\in\, Q_l(S) \\ q \,=\, q_i^S}} \; \sum_{\substack{p \,\in\, P(T, S^{e_i^S}) \\ p \,=\, p_i^T}} \binom{i+j-1}{i} T *_p \Delta_q(S)_1 \otimes \Delta_q(S)_2 \ , \\
\Delta_{\succ}(T \prec S) &= \Delta_{\succ}(T)_1 \otimes \Delta_{\succ}(T)_2 \prec S \ ,
\end{aligned}$$

$$\begin{aligned}
\Delta_{\prec}(T \succ S) &= T \succ \Delta_{\prec}(S)_1 \otimes \Delta_{\prec}(S)_2 \ , \\
\Delta_{\prec}(T \prec S) &= \binom{R_T + L_S - 1}{L_S} T \otimes S + \\
&\quad + \sum_{\substack{q \,\in\, Q_r(T) \\ q \,=\, q_i^T}} \; \sum_{\substack{p \,\in\, P(T^{e_i^T}, S) \\ p \,=\, p_j^S}} \binom{i+j-1}{j} \Delta_q(T)_1 \otimes \Delta_q(T)_2 *_p S \\
&\quad + \sum_{\substack{q \,\in\, Q(S) \\ q \,=\, q_i^S}} \; \sum_{\substack{p \,\in\, P_{>}(T, S^{e_i^S}) \\ p \,=\, p_j^T}} \binom{i+j-1}{i} T *_p \Delta_q(S)_1 \otimes \Delta_q(S)_2
\end{aligned}$$

Proof. The combinatorial description of the products and coproducts in term of shuffle paths and cutting paths gives

$$\Delta_{\succ}(T \succ S) = \sum_{p \in P_{<}(T,S)} \sum_{q \in Q_l(T *_p S)} \Delta_q(T *_p S)$$

Then apply lemma 2.2.3 for the shuffle paths p and cutting paths q. The coefficients appear since for any shuffle path p' obtained from p, and for any cutting path q' obtained from q, such that only the edges $(E(p) \cap E(q)) \cap E(T)$ and $(E(p) \cap E(q)) \cap E(S)$ are shuffled, then, $\Delta_{q'}(T *_{p'} S)$ will give the same element as $\Delta_q(T \prec_p S)$. □

2.3 Dendriform operad and Tamari lattice. The set of planar binary trees can be endowed with the Tamari partial order, see [25], the obtained poset is a lattice known as the Tamari lattice. Chatel and Pons have shown that the intervals of the Tamari lattice are in one-to-one correspondence with interval-posets, see [Theorem 2.8 [7], [8]].

Loday introduced the notion of dendriform operad deeply linked with planar binary trees as explained above. He introduced with Ronco a structure of Hopf algebra on planar binary trees in [18]. The link between Tamari posets and dendriform operad, described by Chapoton, enables us to count elements of some intervals of the Tamari lattice.

Proposition 2.3.1 ([6], prop 3.2)**.** *The dendriform and duplicial products enable the description of the following intervals:*

$$[T \triangleright S; T \triangleleft S] = \{X | X \in T \prec S\},$$

for any planar binary trees T and S.

Let us remark that the interval-poset associated with interval $[T \triangleright S; T \triangleleft S]$ has two connected components, corresponding respectively to T and S. We can now enumerate elements in this type of interval, using Lemma 2.2.4.

Corollary 2.3.2. *The number of elements in the intervals described above is given by:*

$$|[T \triangleright S; T \triangleleft S]| = \binom{R_T + L_S - 1}{L_S}$$

where R_T is the number of vertices on the rightmost path of the planar binary tree T and L_S is the number of vertices on the leftmost path of the planar binary tree T, for any planar binary trees T and S.

3 Skew-duplicial bialgebras, co-skew duplicial dendriform algebras.

In the next section we will define a new type of algebras, which as an operad, has the same underlying graded vector space as the tridendriform operad and has the property of being a set-operad, see Mendez's work [21]. The dendriform and duplicial operad share the same property. Moreover it will also verify the same compatibility towards composition as the dendriform and duplicial operads do, namely the operadic composition of the dendriform operad γ_{Dend} and the duplicial composition γ_{Dup} admit the existence of a map π such that the diagram commute :

$$\begin{tikzcd} Dup \circ Dup = Dend \circ Dend \arrow[r, "\gamma_{Dend}"] \arrow[d, "\gamma_{Dup}"'] & Dend \arrow[dl, dashed, "\pi"] \\ Dup & \end{tikzcd}$$

In addition, any tridendriform algebra can be seen as a dendriform algebra, see Loday. Unfortunately, there exists no such thing as extending the duplicial algebra in a "terplicial" algebra, i.e. as operads a symmetric set operad sharing the same underlying graded vector space as the tridendriform operad and such that the right and the left products are both associative. The terplicial operad defined in the next section is then an extension of a new operad, named the *Skew-duplicial operad*, generated by two binary products $\blacktriangleleft$ and $|\blacktriangleright$ satisfying for any elements x, y and z:

$$(x \,|\blacktriangleright\, y) \,|\blacktriangleright\, z = x \,|\blacktriangleright\, (y \,|\blacktriangleright\, z)$$
$$(x \,|\blacktriangleright\, y) \blacktriangleleft z = x \,|\blacktriangleright\, (y \blacktriangleleft z)$$
$$(x \blacktriangleleft y) \blacktriangleleft z = x \blacktriangleleft (y \,|\blacktriangleright\, z)$$

Note that these algebras are associative algebras equipped with an extra binary operation satisfying a condition of L algebra introduced by P. Leroux and a condition of dipterous algebra introduced by J.-L. Loday and M. Ronco.

Proposition 3.1.1. *The free skew-duplicial operad have the same underlying module as duplicial and dendriform operads. The operations* $\blacktriangleleft$ *and* $|\blacktriangleright$ *are defined on two planar binary trees* $S = \vee(S_l, S_r)$ *and* $T = \vee(T_l, T_r)$ *by:*

$$T \,|\blacktriangleright\, S = \vee(T \,|\blacktriangleright\, S_l, S_r), \quad \textit{with } T \,|\blacktriangleright\, \emptyset = T$$
$$T \blacktriangleleft S = \vee(T_l, T_r \,|\blacktriangleright\, S), \quad \textit{with } \emptyset \,|\blacktriangleright\, T = T$$

Proof. The proof is done by checking that the defined product is a skew-duplicial product. The freeness of the defined product comes from the freeness of the dendriform product on planar binary trees: indeed the skew-duplicial product is equivalent to associating to the dendriform product on binary trees the leading term for the order $S > T$ if $|S_l| > |T_l|$ or $|S_l| = |T_l|$ and $|S_l| > |T_l|$. $\square$

The dual co-Skew-duplicial coproduct satisfies the following relations:

$$(\Delta_{|\blacktriangleright} \otimes id) \circ \Delta_{|\blacktriangleright} = (id \otimes \Delta_{|\blacktriangleright}) \circ \Delta_{|\blacktriangleright}$$
$$(\Delta_{|\blacktriangleright} \otimes id) \circ \Delta_{\blacktriangleleft} = (id \otimes \Delta_{\blacktriangleleft}) \circ \Delta_{|\blacktriangleright}$$
$$(\Delta_{\blacktriangleleft} \otimes id) \circ \Delta_{\blacktriangleleft} = (id \otimes \Delta_{|\blacktriangleright}) \circ \Delta_{\blacktriangleleft}$$

It is given on planar binary trees by:

$$\Delta_{|\blacktriangleright}(\vee(T_l;T_r)) = T_l \otimes \vee(\emptyset;T_r) + \Delta_{|\blacktriangleright}(T_l)_1 \otimes \vee(\Delta_{|\blacktriangleright}(T_l)_2;T_r)$$
$$\Delta_{\blacktriangleleft}(\vee(T_l;T_r)) = \vee(T_l;\emptyset) \otimes T_r + \vee(T_l;\Delta_{|\blacktriangleright}(T_r)_1) \otimes \Delta_{|\blacktriangleright}(T_r)_2$$

with $\Delta_{\blacktriangleleft}(\vee(T_l;\emptyset)) = \Delta_{|\blacktriangleright}(\vee(\emptyset;T_r)) = 0$.

This operad enables us to introduce a rigidity theorem for co-skew-duplicial dendriform bialgebras, proven using the recursive definitions of products and coproducts:

Proposition 3.1.2 (Rigidity theorem for co-skew-duplicial dendriform bialgebras)**.** *Any connected dendriform co-skew-duplicial bialgebra satisfying the following confluence laws is free and cofree over its primitive elements:*

$$\Delta_{\blacktriangleleft}(T \succ S) = T \succ \Delta_{\blacktriangleleft}(S)_1 \otimes \Delta_{\blacktriangleleft}(S)_2$$
$$\Delta_{\blacktriangleleft}(T \prec S) = T \otimes S + T \prec \Delta_{|\blacktriangleright}(S)_1 \otimes \Delta_{|\blacktriangleright}(S)_2 + \Delta_{\blacktriangleleft}(T)_1 \otimes \Delta_{\blacktriangleleft}(T)_2 * S$$
$$\Delta_{|\blacktriangleright}(T \succ S) = T \otimes S + T * \Delta_{|\blacktriangleright}(S)_1 \otimes \Delta_{|\blacktriangleright}(S)_2 + \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \succ S$$
$$\Delta_{|\blacktriangleright}(T \prec S) = \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \prec S$$

Proposition 3.1.3 (Rigidity theorem for co-skew-duplicial duplicial bialgebras)**.** *Any connected duplicial co-skew-duplicial bialgebra satisfying the following confluence laws is free and cofree over its primitive elements:*

$$\Delta_{\blacktriangleleft}(T \triangleright S) = T \triangleright \Delta_{\blacktriangleleft}(S)_1 \otimes \Delta_{\blacktriangleleft}(S)_2$$
$$\Delta_{\blacktriangleleft}(T \triangleleft S) = \Delta_{\blacktriangleleft}(T)_1 \otimes \Delta_{\blacktriangleleft}(T)_2 \triangleleft S$$
$$\Delta_{|\blacktriangleright}(T \triangleright S) = T \otimes S + T \triangleright \Delta_{|\blacktriangleright}(S)_1 \otimes \Delta_{|\blacktriangleright}(S)_2 + \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \triangleright S$$
$$\Delta_{|\blacktriangleright}(T \triangleleft S) = \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \triangleleft S$$

Proposition 3.1.4 (Rigidity theorem for co-skew-duplicial skew-duplicial bialgebras)**.** *Any connected skew-duplicial co-skew-duplicial bialgebra satisfying the following confluence laws is free and cofree over its primitive elements:*

$$\Delta_{\blacktriangleleft}(T \blacktriangleleft S) = T \otimes S + \Delta_{\blacktriangleleft}(T)_1 \otimes \Delta_{\blacktriangleleft}(T)_2 \;|\blacktriangleright\; S + T \blacktriangleleft \Delta_{|\blacktriangleright}(S)_1 \otimes \Delta_{|\blacktriangleright}(S)_2$$
$$\Delta_{\blacktriangleleft}(T \;|\blacktriangleright\; S) = T \;|\blacktriangleright\; \Delta_{\blacktriangleleft}(S)_1 \otimes \Delta_{\blacktriangleleft}(S)_2$$
$$\Delta_{|\blacktriangleright}(T \blacktriangleleft S) = \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \blacktriangleleft S$$
$$\Delta_{|\blacktriangleright}(T \;|\blacktriangleright\; S) = T \otimes S + \Delta_{|\blacktriangleright}(T)_1 \otimes \Delta_{|\blacktriangleright}(T)_2 \;|\blacktriangleright\; S + T \;|\blacktriangleright\; \Delta_{|\blacktriangleright}(S)_1 \otimes \Delta_{|\blacktriangleright}(S)_2$$

4 Application: freeness of algebras as dendriform algebras

We will consider two well-known combinatorial Hopf algebras : the algebra of surjections and the algebra of Parking functions. To illustrate our above definitions, we will endow them with their usual dendriform structure and describe a co-duplicial structure or co-skew-duplicial structure verifying the confluence law 1.3.1 or 3.1.4 respectively. As a corollary it will reprove their freeness as a dendriform algebra, see [3, 10, 23, 27, 26] for previous proofs.

4.1 The algebra of surjections : Let us consider the vector space of surjection and recall its dendriform structure. We will denote it as **ST** the Solomon-Tits algebra. It is also denoted FQSym by some authors.

Consider the set $\mathbf{ST}_n^r := \{x : [n] \to [r], x \text{ surjective}\}$ and the vector space $\mathbf{ST} = \oplus_{n \geq r \geq 1} \mathbb{K}[ST_n^r]$. For $x \in \mathbf{ST}_n^r$, we write $x = (x(1), \dots, x(n))$, listing its images, and $r = \max\{x(i), 1 \leq i \leq n\}$. For $x \in \mathbf{ST}_n^r, y \in \mathbf{ST}_m^s$ denote the shifted concatenation by $x \times y = (x(1), \dots, x(n), y(1)+r, \dots, y(m)+r)$.

This vector space can be endowed with a dendriform structure $(\prec, \succ)$, see for example [23, 10, 2], as follows: let $x \in \mathbf{ST}_n^r$, $y \in \mathbf{ST}_m^s$ define

(1) $x \succ y := \sum_{f \in Sh^{\succ}(r,s)} f \circ (x \times y)$,
(2) $x \prec y := \sum_{f \in Sh^{\prec}(r,s)} f \circ (x \times y)$,

where, $f \in Sh^{\prec}$ is a (r,s)-shuffle such that $f(r) > f(r+s)$ and $f \in Sh^{\succ}$ is a (r,s)-shuffle such that $f(r) < f(r+s)$. Fix that $x \succ 1_{\mathbb{K}} := 0 =: 1_{\mathbb{K}} \prec x$ and $x \prec 1_{\mathbb{K}} := x =: 1_{\mathbb{K}} \succ x$, for all $x \in \mathbf{ST}$.

Note that **ST** as a dendriform algebra is in bijection with levelled trees as proven in [18]. The coduplicial structure we consider can be understood on trees as follows : Let T be a levelled tree, $\Delta_{\triangleright}(T) = T_1 \otimes T_2$ T_1, T_2 are obtained by cutting on the first of the tree, and $\Delta_{\triangleleft}(T) = T^1 \otimes T^2$ where where T^1, T^2 are obtained by cutting on the last edge of the tree.

Proposition 4.1.1. **ST** *endowed with the coproducts* $\Delta_{\triangleright}, \Delta_{\triangleleft}$ *and the dendriform structure* $(\prec, \succ)$ *is a* Dup^c*-Dend bialgebra.*

In [26, 3.2.1, proposition 4], Vong introduces on ST the following operations : $x \mid\!\blacktriangleright y = x \times y$, $x \blacktriangleleft y = (x(1), \dots, x(n-1), y(1)+r, \dots, y(m)+r, x(n))$ with $x \in ST_n^r$, $y \in ST_m^s$. These operations verify the skew-duplicial relations. Dualising them give co-duplicial cooperations.

Proposition 4.1.2. ST *endowed with the coproducts* $(\Delta_{\blacktriangleleft}, \Delta_{|\blacktriangleright})$ *and the dendriform structure* $(\prec, \succ)$ *is a co-skew-duplicial dendriform bialgebra.*

Corollary 4.1.3. [10, 3, 23, 27] **ST** *is free as a dendriform algebra on its primitives.*

It is straightforward from theorem 1.3.3 with the co-duplicial dendriform structure or theorem 3.1.4 with the co-skew duplicial dendriform structure.

Remark 4.1.4. The basis of the primitives, in the case of the coduplicial dendriform bialgebra structure, is not the same as in [3] as in dimension 3 the primitives are $(1,3,2), (2,3,1)$ whereas in [3] it is $(1,2,1), (2,3,1)$. The number of elements of ST for each dimension is given by the Fubini numbers [24, A00670].

Remark 4.1.5. In the case of the co-skew duplicial dendriform bialgebra structure, this corollary gives an algebraic rewriting of Vong's proof [26, section 3], which uses reductions and Grobner basis arguments.

4.2 The Parking function algebra: The set of Parking functions can be endowed with a dendriform structure given by the work of Novelli-Thibon [23]. We follow [2].

Definition 4.2.1. A map $f : [n] \to [n]$ is called a *n-non-decreasing parking function* if $f(i) \leq i$ for $1 \leq i \leq n$. The set of n-non-decreasing parking functions is denoted by $NDPF_n$.The composition $f := f^{\uparrow} \circ \sigma$ of a non-decreasing parking function $f^{\uparrow} \in NDPF_n$ and a permutation $\sigma \in S_n$ is called a *n-parking function*. The set of n-parking functions is denoted by PF_n. The subset of those such that $\max\{f(i), 1 \leq i \leq n\} = r$ is denoted PF_n^r.

Let $\mathbf{PQSym}^*$ denote the vector space spanned by the set $\bigcup_{n\geq 1} PF_n$ of parking functions. The binary operations $\prec$ and $\succ$ on $\mathbf{PQSym}^*$ are defined in a similar way that in the case of $\mathbf{ST}$:

$$f \prec g := \textstyle\sum_{\max(h)>\max(k)} hk, \quad f \succ g := \textstyle\sum_{\max(h)\leq\max(k)} hk,$$

where the sums are taken over all pairs of maps (h, k) verifying that hk is parking, $Park(h) = f$ and $Park(k) = g$, for $f, g \in \bigcup_{n\geq 1} PF_n$ and the map $Park : \bigcup\limits_{n\geq 1} \mathcal{F}_n \longrightarrow \bigcup\limits_{n\geq 1} PF_n$ is the Parking counterpart of the standardisation.

We will now define on the Parking function a structure of coduplicial coalgebra similarly to $\mathbf{ST}$. Consider $f \in \mathrm{P}F_n^r$, a left-to-right maximum i is such that all the images that precede are smaller than $f(i) : f(i-k) < f(i)$, $i-k \geq 0$. A right-to left maximum i verifies $f(i+k) < f(i)$, $i+k \leq n$. We will denote by $LR(f)$ the list of the left-to-right maxima of f and by $RL(f)$ its list of right-to-left maxima.

For $f \in \mathrm{PF}_n^r$, $\Delta_{\triangleright}(f) = (f(1), \ldots, f(l_k - 1)) \otimes \mathrm{Park}(f(l_k), \ldots, f(n))$ with $l_k \in LR(f) = (l_1, \ldots, l_f)$ the maximum element such that $f(l_k) = f(l_{k+1})+1$ and that f_1 belong to PF.

For $f \in \mathrm{PF}_n^r$: $\Delta_{\triangleleft}(f) = \mathrm{park}(f(1), \ldots, f(r_k)) \otimes (f(r_k+1), \ldots, f(n))$ where $r_k \in RL(f) = (r_1, \ldots, r_f)$ the minimum element such that $f_2 \in \mathrm{PF}$ and that $f(r_k) = f(r_{k+1}) - 1$.

Proposition 4.2.2. $\mathbf{PQSym}^*$ *endowed with the coproducts $\Delta_{\triangleright}, \Delta_{\triangleleft}$ and the dendriform structure $(\prec, \succ)$ is a Dup^c-Dend bialgebra.*

Proof. The coduplicial structure is satisfied as the Parkisation of words preserve the maxima. The confluence law is verified as in $\mathbf{ST}$ by direct inspection. □

As a corollary:

Corollary 4.2.3. [23, 3] *The dendriform algebra of Parking function is free as a dendriform algebra over the vector space of its primitive elements.*

5 Terplicial and tridendriform operads.

5.1 Definitions: tridendriform (co)algebras. Let us recall the relations governing a q-tridendriform algebra linking the tridendriform structure described in [18] for $q = 1$ and in [5] for $q = 0$:

Definition 5.1.1. [2] A *q-tridendriform algebra* is a vector space A together with three operations $\prec : A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ : A \otimes A \to A$, satisfying the following relations:

$$
\begin{array}{llll}
(a \prec b) \prec c & = a \prec (b \prec c + b \succ c + q\ b \cdot c), & & \\
a \succ (b \succ c) & = (a \prec b + a \succ b + q\ a \cdot b) \succ c, & & \\
(a \succ b) \prec c & = a \succ (b \prec c), & (a \cdot b) \cdot c & = a \cdot (b \cdot c), \\
(a \succ b) \cdot c & = a \succ (b \cdot c), & (a \prec b) \cdot c & = a \cdot (b \succ c), \\
(a \cdot b) \prec c & = a \cdot (b \prec c). & &
\end{array}
$$

Note that the operation $* := \prec + q\ \cdot + \succ$ is associative. Moreover, given a q-tridendriform algebra $(A, \prec, \cdot, \succ)$, the space A equipped with the binary operations $\prec$ and $\overline{\succ} := q \cdot + \succ$ is a dendriform algebra.

Example 5.1.2. Let T_n be the set of all planar reduced rooted trees with $n+1$ leaves. Denote by $T_\infty = \bigcup_n T_n$. Any $t \in T_n$ may be written in a unique way as $t = \bigvee(t^1, \dots, t^r)$, with $t^i \in T_{n_i}$ and $\sum_{i=1}^r n_i + r - 1 = n$.

On the space $\mathbb{K}[T_\infty]$, define operations $\succ$, $\cdot$ and $\prec$ recursively as follows:

$$
\begin{aligned}
t \succ | = t \cdot | = | \cdot t = | \prec t = 0, &\ \text{ for all } t \in T_\infty, \\
| \succ t = t \prec | = t, &\ \text{ for all } t \in T_\infty, \\
t \prec w &:= \bigvee(t^1, \dots, t^{r-1}, t^r * w), \\
t \cdot w &:= \bigvee(t^1, \dots, t^{r-1}, t^r * w^1, w^2, \dots, w^l), \\
t \succ w &:= \bigvee(t * w^1, w^2, \dots, w^l),
\end{aligned}
$$

for $t = \bigvee(t^1, \dots, t^r)$ and $w = \bigvee(w^1, \dots, w^l)$, where $*$ is the associative product $* = \succ + q \cdot + \prec$ previously defined.

Following [5] and [19], $(\mathbb{K}[T_\infty], \succ, \cdot, \prec)$ is the free q-tridendriform algebra spanned by the unique element of T_1.

Definition 5.1.3. A *q-tridendriform coalgebra*, or co-tridendriform coalgebra, is a vector space V endowed with three coproducts $\Delta_\prec, \Delta_., \Delta_\succ$ satisfying the following relations:

$$
\begin{array}{ll}
(\Delta_\prec \otimes id) \circ \Delta_\prec = (id \otimes \Delta_*) \circ \Delta_\prec\ , & (\Delta_\succ \otimes id) \circ \Delta_\prec = (id \otimes \Delta_\prec) \circ \Delta_\succ\ , \\
(\Delta_* \otimes id) \circ \Delta_\succ = (id \otimes \Delta_\succ) \circ \Delta_\succ\ , & (id \otimes \Delta_.) \circ \Delta_. = (\Delta_. \otimes id) \circ \Delta_.\ , \\
(\Delta_\succ \otimes id) \circ \Delta_. = (id \otimes \Delta_.) \circ \Delta_\succ\ , & (\Delta_\prec \otimes id) \circ \Delta_. = (id \otimes \Delta_\succ) \circ \Delta_.\ , \\
(\Delta_. \otimes id) \circ \Delta_\prec = (id \otimes \Delta_\prec) \circ \Delta_.\ . &
\end{array}
$$

where $\Delta_* = \Delta_\prec + q\Delta_. + \Delta_\succ$.

A co-augmented *conilpotent* tridendriform coalgebra C is a coalgebra verifying that: $C = \cup_{n \geq 0} F_n C$ where $F_0 C = \mathbb{K}$, $F_1 C = \{x \in C \mid \Delta_\prec(x) = \Delta_\succ(x) = \Delta_.(x) = 0\}$, $F_n C = \{x \in C \mid \Delta_\prec(x) \in F_{n-1} C^{\otimes 2}, \Delta_\succ(x) \in F_{n-1} C^{\otimes 2}, \Delta_.(x) \in F_{n-1} C^{\otimes 2}\}$.

Example 5.1.4. As in the dendriform framework, we will consider the coproducts obtained as duals of the above tridendriform products on the vector space generated by planar reduced rooted trees $\mathbb{K}[T_\infty]$.The isomorphism between $\mathbb{K}[T_\infty]$ and its graded dual $\mathbb{K}[T_\infty]^*$ given by the basis of planar rooted trees induces a cotridendriform coalgebra structure on $\mathbb{K}[T_\infty]$: for every $T \in T_\infty$ the cooperations are given by $\Delta_\bullet(T) = \sum T_{(\bullet 1)} \otimes T_{(\bullet(2)}$ where the sum runs over all pairs $(T_{(\bullet 1)}, T_{(\bullet 2)})$ such that $T^*(T_{(\bullet 1)} \bullet T_{(\bullet 2)}) \neq 0$, with $\bullet = \prec, \succ$ or $\cdot$. The definition of the operations gives a constructive way to define the cooperations as:

$$\Delta_\succ(\vee(t_1,\dots,t_n)) = \Delta_*(t_1)_{(*1)} \otimes \vee(\Delta_*(t_1)_{(*2)}, t_2,\dots,t_n) + t_1 \otimes \vee(\emptyset, t_2,\dots,t_n)$$

$$\begin{aligned}\Delta_\cdot(\vee(t_1,\dots,t_n)) = &\sum_{i=2}^{n-1} \vee(t_1,\dots,t_{i-1},\Delta_*(t_i)_{(*1)}) \otimes \vee(\Delta_*(t_i)_{(*2)}, t_{i+1},\dots,t_n)\\ &+ \vee(t_1,\dots,t_{i-1},\emptyset) \otimes \vee(t_i,\dots,t_n) + \vee(t_1,\dots,t_i) \otimes \vee(\emptyset, t_{i+1},\dots,t_n)\end{aligned}$$

$$\Delta_\prec(\vee(t_1,\dots,t_n)) = \vee(t_1,\dots,t_{n-1},\Delta_*(t_n)_{(*1)}) \otimes \Delta_*(t_n)_{(*2)} + \vee(t_1,\dots,t_{n-1},\emptyset) \otimes t_n\ ,$$

where $\Delta_* = \Delta_\prec + q\Delta_\cdot + \Delta_\succ$.

This definition gives well-defined coproducts verifying the cotridendriform relations. Note that any free tridendriform algebra is naturally endowed with this dual coalgebra structure.

Dualising the proof for the freeness of T_∞ as a tridendriform algebra, one gets that this structure on T_∞ is the cofree conilpotent tridendriform coalgebra.

5.2 Definitions: terplicial (co)algebras. From the tridendriform operad, we define a new set-operad called terplicial, on which the tridendriform operad is quasi-set (see [1]), by analogy with the pair (Dend, skew-dupl). It is to be noted that an analogue of the pair (Dend, Dupl) is not possible as the analogue of Dupl with three associative products cannot be defined.

Definition 5.2.1. A *terplicial algebra* is a vector space V endowed with three binary products $\{\triangleleft, \triangledown, \shortmid\!\triangleright\}$ satisfying the following relations:

$$\begin{gathered}\shortmid\!\triangleright \text{ and } \triangledown \text{ are associative,}\\ (x \triangleleft y) \triangleleft z = x \triangleleft (y \shortmid\!\triangleright z)\\ (x \shortmid\!\triangleright y) \triangleleft z = x \shortmid\!\triangleright (y \triangleleft z)\\ (x \triangledown y) \triangleleft z = x \triangledown (y \triangleleft z)\\ (x \shortmid\!\triangleright y) \triangledown z = x \shortmid\!\triangleright (y \triangledown z)\\ (x \triangleleft y) \triangledown z = x \triangledown (y \shortmid\!\triangleright z)\end{gathered}$$

All the equations but the second and the last coincide with relations satisfied by triduplicial algebra defined by J.-C. Novelli and J.-Y. Thibon in [22].

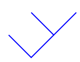
Consider the planar rooted trees, and denote by $\vee(t_1,\dots,t_n)$ a planar tree $T \in T_\infty$ whose root has arity n and such that the t_i are the (possibly empty) subtrees of T rooted in the children of the root of T. We can now describe the free terplicial algebras.

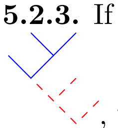
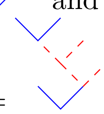
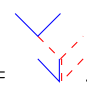
Theorem 5.2.2. *The free terplicial algebra on a vector space V can be described as the algebra whose underlying vector space has a basis given by reduced planar rooted trees with leaves decorated by V: $Ter(V) = \oplus \mathbb{K}[T_n] \otimes V^{\otimes n}$. Hence, the dimension of the space of operations of arity n in the triplicial operad is given by the Schroeder-Hipparchus number.*

The operations $\triangleleft$, $\shortmid\!\triangleright$ and $\triangledown$ on free terplicial algebras are described recursively as follows, for any tree $T = \vee(t_1, \dots, t_n)$ and $S = \vee(s_1, \dots, s_m)$, and denoting by $\emptyset$ the empty tree:

$$\begin{aligned}
T \triangleleft S &= \vee(t_1, \dots, t_{n-1}, t_n \shortmid\!\triangleright S) \\
T \triangledown S &= \vee(t_1, \dots, t_{n-1}, t_n \shortmid\!\triangleright s_1, s_2, \dots, s_m) \\
T \shortmid\!\triangleright S &= \vee(T \shortmid\!\triangleright s_1, s_2 \dots, s_m),
\end{aligned}$$

given that, $\emptyset \shortmid\!\triangleright T = T$ and $T \shortmid\!\triangleright \emptyset = T$.

Example 5.2.3. If $T =$  and $S =$ , the products are given by:

$T \shortmid\!\triangleright S =$ , $T \triangleleft S =$  and $T \triangledown S =$ .

Proof. T_∞ endowed with these operations satisfy the terplicial relations, see section 7.4.2.

The universal property of terplicial algebras is verified: for any morphism from $f : V \to A$, with A a terplicial algebra, $\iota : V \to Ter(V)$ the canonical injection, then there exists a unique terplicial morphism $\phi : Ter(V) \to A$ defined as

$$\begin{aligned}
&\vee(t_1, \dots, t_n) \otimes v_1 \dots v_{k_1} v_{k_1+1} \dots v_{k_1+\dots+k_n} \mapsto \\
&\phi(t_1 \otimes v_1 \dots v_{k_1}) \shortmid\!\triangleright \\
&(\phi(\vee(\emptyset, t_2) \otimes 1 v_{k_1+1} \dots v_{k_2}) \triangledown \dots \triangledown \phi(\vee(\emptyset, t_{n-1}) \otimes 1 v_{k_{n-2}+1} \dots v_{k_{n-1}}) \triangledown \phi(\vee(\emptyset, \emptyset) \otimes 1 \cdot 1)) \\
&\triangleleft \phi(t_n \otimes v_{k_{n-1}+1 \dots k_n}) \ .
\end{aligned}$$

The uniqueness of the morphism is obtained by construction. $\square$

Dualising the notion of terplicial algebras, into terplicial coalgebras gives:

Definition 5.2.4. A *terplicial coalgebra*, or coterplicial coalgebra, is a vector space C endowed with three coproducts $\Delta_\triangleleft, \Delta_\triangledown, \Delta_\triangleright : C \to C \otimes C$ satisfying the following relations:

$$\begin{aligned}
&\Delta_{\triangleright} \text{ and } \Delta_{\triangledown} \text{ are co-associative,}\\
&(\Delta_{\triangleleft}\otimes id)\circ\Delta_{\triangleleft} = (id\otimes\Delta_{\triangleright})\circ\Delta_{\triangleleft}\ ,\\
&(\Delta_{\triangleright}\otimes id)\circ\Delta_{\triangleleft} = (id\otimes\Delta_{\triangleleft})\circ\Delta_{\triangleright}\ ,\\
&(\Delta_{\triangledown}\otimes id)\circ\Delta_{\triangleleft} = (id\otimes\Delta_{\triangleleft})\circ\Delta_{\triangledown}\ ,\\
&(\Delta_{\triangleright}\otimes id)\circ\Delta_{\triangledown} = (id\otimes\Delta_{\triangledown})\circ\Delta_{\triangleright}\ ,\\
&(\Delta_{\triangleleft}\otimes id)\circ\Delta_{\triangledown} = (id\otimes\Delta_{\triangleright})\circ\Delta_{\triangledown}\ .
\end{aligned}$$

A co-augmented *conilpotent* terplicial coalgebra C is a coalgebra verifying that: $C = \cup_{n\geq 0} F_n C$ where $F_0C = \mathbb{K}$, $F_1C = \{x \in C \mid \Delta_{\triangleleft}(x) = \Delta_{\triangledown}(x) = \Delta_{\triangleright}(x) = 0\}$, $F_nC = \{x \in C \mid \Delta_{\triangleleft}(x) \in F_{n-1}C^{\otimes 2}, \Delta_{\triangledown}(x) \in F_{n-1}C^{\otimes 2}, \Delta_{\triangleright}(x) \in F_{n-1}C^{\otimes 2}\}$.

Example 5.2.5. We introduce the dual coproduct associated to the products $\triangleleft$, $\triangledown$ and $\shortmid\!\triangleright$. They are given inductively on $T = \vee(t_1, \dots, t_n)$ by:

- if $t_1 = \emptyset$, $\Delta_{\triangleright}(T) = 0$
- if $t_n = \emptyset$, $\Delta_{\triangleleft}(T) = 0$
- if $n \leq 2$, $\Delta_{\triangledown}(T) = 0$

and, otherwise,

$$\Delta_{\triangleright}(T) = t_1 \otimes \vee(\emptyset, \dots, t_n) + \Delta_{\triangleright}(t_1)_1 \otimes \vee(\Delta_{\triangleright}(t_1)_2, \dots, t_n),$$

$$\begin{aligned}\Delta_{\triangledown}(T) = \sum_{i=2}^{n-1} &\vee(t_1,\dots,t_{i-1},\emptyset)\otimes\vee(t_i,\dots,t_n) + \vee(t_1,\dots,t_i)\otimes\vee(\emptyset,t_{i+1},\dots,t_n)\\ &+ \vee(t_1,\dots,t_{i-1},\Delta_{\triangleright}(t_i)_1)\otimes\vee(\Delta_{\triangleright}(t_i)_2,t_{i+1},\dots,t_n)\ ,\end{aligned}$$

$$\Delta_{\triangleleft}(T) = \vee(t_1, \dots, t_{n-1}, \emptyset) \otimes t_n + \vee(t_1, \dots, t_{n-1}, \Delta_{\triangleright}(t_n)_1) \otimes \Delta_{\triangleright}(t_n)_2\ .$$

It is by duality, the cofree conilpotent terplicial coalgebra on one generator, which can be extended to the cofree conilpotent terplicial coalgebra on generators by decorating the leaves, in a similar manner as the free tridendriform algebras.

5.3 Rigidity theorem for bialgebras endowed with terplicial and tridendriform structures

5.3.1 Terplicial bialgebras

Definition 5.3.1. A *terplicial bialgebra* is a vector space $\mathcal{H}$ endowed with a terplicial algebra structure $(\mathcal{H}, \shortmid\!\triangleright, \triangledown, \triangleleft)$ and a co-terplicial coalgebra structure $(\mathcal{H}, \Delta_{\triangleright}, \Delta_{\triangledown}, \Delta_{\triangleleft})$ satisfying the following mixed ditributive laws:

$$\begin{aligned}
\Delta_{\triangleright}(T \shortmid\!\triangleright S) =& T\otimes S + \Delta_{\triangleright}(T)_1\otimes\Delta_{\triangleright}(T)_2 \shortmid\!\triangleright S + T \shortmid\!\triangleright \Delta_{\triangleright}(S)_1\otimes\Delta_{\triangleright}(S)_2\\
\Delta_{\triangleright}(T\triangledown S) =& \Delta_{\triangleright}(T)_1\otimes\Delta_{\triangleright}(T)_2\triangledown S\\
\Delta_{\triangleright}(T\triangleleft S) =& \Delta_{\triangleright}(T)_1\otimes\Delta_{\triangleright}(T)_2\triangleleft S\\
\Delta_{\triangledown}(T \shortmid\!\triangleright S) =& T \shortmid\!\triangleright\Delta_{\triangledown}(S)_1\otimes\Delta_{\triangledown}(S)_2
\end{aligned}$$

$$
\begin{aligned}
\Delta_{\triangledown}(T \triangledown S) =& T \otimes S + \Delta_{\triangledown}(T)_1 \otimes \Delta_{\triangledown}(T)_2 \triangledown S + T \triangledown \Delta_{\triangledown}(S)_1 \otimes \Delta_{\triangledown}(S)_2 \\
& + \Delta_{\triangleleft}(T) \otimes \Delta_{\triangleleft}(T)_2 \shortmid\!\triangleright S + T \triangleleft \Delta_{\triangleright}(S)_1 \otimes \Delta_{\triangleright}(S)_2 \\
\Delta_{\triangledown}(T \triangleleft S) =& \Delta_{\triangledown}(T)_1 \otimes \Delta_{\triangledown}(T)_2 \triangleleft S \\
\Delta_{\triangleleft}(T \shortmid\!\triangleright S) =& T \shortmid\!\triangleright \Delta_{\triangleleft}(S)_1 \otimes \Delta_{\triangleleft}(S)_2 \\
\Delta_{\triangleleft}(T \triangledown S) =& T \triangledown \Delta_{\triangleleft}(S)_1 \otimes \Delta_{\triangleleft}(S)_2 \\
\Delta_{\triangleleft}(T \triangleleft S) =& T \otimes S + \Delta_{\triangleleft}(T)_1 \otimes \Delta_{\triangleleft}(T)_2 \shortmid\!\triangleright S + T \triangleleft \Delta_{\triangleright}(S)_1 \otimes \Delta_{\triangleright}(S)_2 \ .
\end{aligned}
$$

Proposition 5.3.2. *These above relations are a confluence law.*

The only property that has to be checked is the compatibility with the operad structure. This comes from the fact that the relations are computed on planar rooted trees. The proof is postponed to the section 7.4.2 on the combinatorial description of the products and coproducts.

Applying [1, corollary 2.1.4] :

Proposition 5.3.3 (Rigidity theorem for coterplicial terplicial bialgebras)**.** *Any connected terplicial co-terplicial bialgebra is free and cofree over its primitive elements.*

The bialgebra T_∞ is then free as a terplicial algebra and cofree as a connected terplicial coalgebra over one element.

5.3.2 Co-terplicial tridendriform bialgebras

Definition 5.3.4. A *co-terplicial tridendriform bialgebra* is a vector space $\mathcal{H}$ endowed with a terplicial coalgebra structure $(\mathcal{H}, \Delta_{\triangleright}, \Delta_{\triangledown}, \Delta_{\triangleleft})$, a tridendriform algebra structure $(\mathcal{H}, \prec, \cdot, \succ)$ satisfying the following mixed ditributive laws:

$$
\begin{aligned}
\Delta_{\triangleright}(T \prec S) =& (\Delta_{\triangleright}(T))_1 \otimes (\Delta_{\triangleright}(T))_2 \prec S \\
\Delta_{\triangleright}(T \cdot S) =& (\Delta_{\triangleright}(T))_1 \otimes (\Delta_{\triangleright}(T))_2 \cdot S \\
\Delta_{\triangleright}(T \succ S) =& T \otimes S + (\Delta_{\triangleright}(T))_1 \otimes (\Delta_{\triangleright}(T))_2 \succ S + T * (\Delta_{\triangleright}(S))_1 \otimes (\Delta_{\triangleright}(S))_2 \\
\Delta_{\triangledown}(T \prec S) =& (\Delta_{\triangledown}(T))_1 \otimes (\Delta_{\triangledown}(T))_2 \prec S \\
\Delta_{\triangledown}(T \cdot S) =& T \otimes S + (\Delta_{\triangledown}(T))_1 \otimes (\Delta_{\triangledown}(T))_2 \cdot S + T \cdot (\Delta_{\triangledown}(S))_1 \otimes (\Delta_{\triangledown}(S))_2 + \\
& (\Delta_{\triangleleft}(T))_1 \otimes (\Delta_{\triangleleft}(T))_2 \succ S + T \prec (\Delta_{\triangleright}(S))_1 \otimes (\Delta_{\triangleright}(S))_2 \\
\Delta_{\triangledown}(T \succ S) =& T \succ (\Delta_{\triangledown}(S))_1 \otimes (\Delta_{\triangledown}(S))_2 \\
\Delta_{\triangleleft}(T \prec S) =& T \otimes S + (\Delta_{\triangleleft}(T))_1 \otimes (\Delta_{\triangleleft}(T))_2 * S + T \prec (\Delta_{\triangleright}(S))_1 \otimes (\Delta_{\triangleright}(S))_2 \\
\Delta_{\triangleleft}(T \cdot S) =& T \cdot (\Delta_{\triangleleft}(S))_1 \otimes (\Delta_{\triangleleft}(S))_2 \\
\Delta_{\triangleleft}(T \succ S) =& T \succ (\Delta_{\triangleleft}(S))_1 \otimes (\Delta_{\triangleleft}(S))_2
\end{aligned}
$$

where $* = \succ + \prec + \cdot$.

Proposition 5.3.5. *The relations introduced in the previous definition are a confluence law.*

The property that has to be checked is the compatibility with the operadic structure. This comes from the fact that the these relations are computed on rooted planar trees. The proof of these relations are then postponed to section 7.4.2 using the combinatorial description of the terplicial products and coproducts.

We apply the result of [1, corollary 2.1.4] :

Proposition 5.3.6 (Rigidity theorem for coterplicial tridendriform bialgebras)**.** *Any connected tridendriform co-terplicial bialgebra is free and cofree over its primitive elements.*

Therefore the bialgebra T_∞ is free as a tridendriform algebra on one element, and cofree as a connected co-terplicial algebra on one element.

5.3.3 Tridendriform bialgebras

Tridendriform-tridendriform bialgebras: Let us denote $\Delta^1_\bullet = \Delta_\bullet$, $\Delta^{k+1}_\bullet = (id^k \otimes \Delta_\bullet) \circ \Delta^k_\bullet$, and $\bullet^1 = \bullet$, $\bullet^{k+1} = \bullet^k \circ (id^k \otimes \bullet)$, where $\bullet$ stands for one of the symbols $\prec, \cdot$ or $\succ$.

For the sake of readability, we will denote $\alpha_\prec := \Delta_\prec + \sum_{k\geq 1}(-1)^k(id\otimes \prec) \circ \Delta^{k+1}_\prec$, $\alpha_\succ = \Delta_\succ + \sum_{k\geq 1}(-1)^k(\succ \otimes id^k) \circ \Delta^{k+1}_\succ$.

Definition 5.3.7. A *co-tridendriform tridendriform bialgebra* is a vector space $\mathcal{H}$ endowed with a co-tridendriform coalgebra structure $(\mathcal{H}, \Delta_\succ, \Delta_., \Delta_\prec)$ and a tridendriform algebra structure $(\mathcal{H}, \succ, \cdot, \prec)$ linked by the confluence laws given by: for any $x, y \in \mathcal{H}$

$$\Delta_\succ(x \succ y) = \Delta_*(x * \alpha_\succ(y)_1)_1 \otimes \Delta_*(x * \alpha_\succ(y)_1)_2 \succ \alpha_\succ(y)_2 + x * \alpha_\succ(y)_1 \otimes \alpha_\succ(y)_2$$

$$\Delta_\succ(x \cdot y) = \Delta_\succ(x)_1 \otimes \Delta_\succ(x)_2 \cdot y$$

$$\Delta_\succ(x \prec y) = \Delta_\succ(x)_1 \otimes \Delta_\succ(x)_2 \prec y$$

$$\Delta_.(x \succ y) = x \succ \Delta_.(y)_1 \otimes \Delta_.(y)$$

$$\Delta_.(x \cdot y) = \Delta_.(x) \otimes \Delta_.(x)_2 \cdot y + x \cdot \Delta_.(y)_1 \otimes \Delta_.(y)_2 +$$
$$+ \alpha_\prec(x)_1 \prec (\Delta_*(\alpha_\prec(x)_2 * \alpha_\succ(y)_1)_1) \otimes (\Delta_*(\alpha_\prec(x)_2 * \alpha_\succ(y)_1)_2) \succ \alpha_\succ(y)_2$$

$$\Delta_.(x \prec y) = \Delta_.(x)_1 \otimes \Delta_.(x)_2 \prec y$$

$$\Delta_\prec(x \succ y) = x \succ \Delta_\prec(y)_1 \otimes \Delta_\prec(y)_2$$

$$\Delta_\prec(x \cdot y) = x \cdot \Delta_\prec(y)_1 \otimes \Delta_\prec(y)_2$$

$$\Delta_\prec(x \prec y) = \alpha_\prec(x)_1 \prec \Delta_*(\alpha_\prec(x)_2 * y)_1 \otimes \Delta_*(\alpha_\prec(x)_2 * y)_2 + \alpha_\prec(x)_1 \otimes \alpha_\prec(x)_2 * y$$

The definition is considered for $q = 1$ but can be extended for any q.

Proposition 5.3.8. *The above relations are a confluence law.*

Proof. The only property that has to be checked is the compatibility with the operadic structure. This comes from the fact that these relations are computed on planar rooted trees. Prove by induction that for any tree $T = \vee(t_1, \ldots, t_n)$ the element $\alpha_\succ(T)$ is equal to $t_1 \otimes \vee(\emptyset, t_2, \ldots, t_n)$, respectively, $\alpha_\prec(T)$ is equal to $\vee(t_1, t_2, \ldots, t_{n-1}, \emptyset) \otimes t_n$. The induction is on the cofiltration with respect to $\Delta_\succ, \Delta_\prec$ respectively. Moreover $\alpha_. = \Delta_. - (\prec \circ\alpha_\prec \otimes \succ \circ\alpha_\succ) \circ \Delta_.$ applied to T is given by $\sum_{i=2}^{n-1} \vee(t_1, \ldots, t_i) \otimes \vee(\emptyset, t_{i+1}, \ldots, t_n) + \vee(t_1, \ldots, t_{i-1}, \emptyset) \otimes \vee(t_i, t_2, \ldots, t_n)$. $\square$

Applying [1, corollary 2.1.4] :

Theorem 5.3.9 (Rigidity theorem for co-tridendriform-tridendriform bialgebras)**.** *Any connected co-tridendriform tridendriform bialgebra is free and cofree over its primitives.*

6 Application to the freeness of some tridendriform algebras

6.1 Application to the freeness of the Solomon-Tits algebra as a tridendriform algebra. The Solomon-Tits algebra can be endowed with a tridendriform structure, see for example [23, 2]. We keep the notations taken in section 4.1.

The concatenation product $\times : \mathbf{ST}_n^r \otimes \mathbf{ST}_m^s \longrightarrow \mathbf{ST}_{n+m}^{r+s}$ is given by the formula:

$$f \times g := (f(1), \ldots, f(n), g(1) + r, \ldots, g(m) + r).$$

Similarly, for $K = \{j_1 < \cdots < j_l\} \subseteq \{1, \ldots, r\}$, the co-restriction of x to K is denoted $x|^K := \mathrm{std}(x(s_1), \ldots, x(s_q))$, for $x^{-1}(K) = \{s_1 < \cdots < s_q\}$.

For an element $x \in \mathbf{ST}_n^r$, we denote by $\lambda(x)$ the cardinal of $x^{-1}(\{r\})$.

Suppose that $x^{-1}(r) = \{j_1 < \cdots < j_{\lambda(x)}\}$, and let $x' \in \mathbf{ST}_{n-k}^{r-1}$ be the co-restriction $x' := x|^{\{1,\ldots,r-1\}}$. We denote x as $x = \prod_{j_1<\cdots<j_{\lambda(x)}} x'$.

Let $(n_1, \ldots, n_p)$ be a composition of n. An element in $f \in \mathbf{ST}_n$ is a $(n_1, \ldots, n_p)$-*stuffle* if

$$f(n_1 + \cdots + n_i) < f(n_1 + \cdots + n_i + 1) < \cdots < f(n_1 + \cdots + n_i + n_{i+1}),$$

for $0 \leq i \leq p - 1$.

We denote by $SH(n_1, \ldots, n_p)$ the set of all $(n_1, \ldots, n_p)$-stuffles.

For a composition $(n_1, \ldots, n_p)$ of n, we denote:

(1) $\mathrm{SH}^\prec(n_1, \ldots, n_p)$ the subset of all surjective maps $f \in \mathrm{SH}(n_1, \ldots, n_p)$ such that $f(n_1) > f(n_1 + n_2) > \cdots > f(n)$.
(2) $\mathrm{SH}^\succ(n_1, \ldots, n_p)$ the subset of all surjective maps $f \in \mathrm{SH}(n_1, \ldots, n_p)$ such that $f(n_1) < f(n_1 + n_2) < \cdots < f(n)$.
(3) $\mathrm{SH}^\bullet(n_1, \ldots, n_p)$ the subset of all surjective maps $f \in \mathrm{SH}(n_1, \ldots, n_p)$ such that $f(n_1) = f(n_1 + n_2) = \cdots = f(n)$.

Let $x \in \mathbf{ST}_n^r, y \in \mathbf{ST}_m^s$, the tridendriform structure on $\mathbf{ST}$ is defined as follows:

$$x \succ y := \sum_{f \in SH^\succ(r,s)} f \circ (x \times y), \qquad x \cdot y := \sum_{f \in SH^\bullet(r,s)} f \circ (x \times y),$$
$$x \prec y := \sum_{f \in SH^\prec(r,s)} f \circ (x \times y).$$

The work of Vong [27] can be understood as a construction of a terplicial algebraic structure on $\mathbf{ST}^s_m$. For $x \in \mathbf{ST}^r_n, y \in \mathbf{ST}^s_m$ define the operations

$$x \triangleleft y = \prod_{j_1^x,\dots,j^x_{\lambda(x)}} x' \times y \qquad x \triangledown y = \prod_{j_1^x,\dots,j^x_{\lambda(x)},j_1^y,\dots,j^y_{\lambda(y)}} x' \times y'$$

$$x \shortmid\!\triangleright y = x \times y$$

where $x = \prod_{j_1^x,\dots,j^x_{\lambda(x)}} x'$ and $y = \prod_{j_1^y,\dots,j^y_{\lambda(x)}} y'$.

The relations are checked by direct inspection.

In this section, we focus on proving the freeness of **ST** as a free tridendriform algebra and terplicial algebra by adding a co-terplicial structure to **ST** which is dual to the terplicial structure. The confluence laws are those introduced above. This viewpoint permits to give a way to understand the double application of Grœbner basis algorithm as seen in the work of Vong as terplicial-tridendriform or terplicial-terplicial bialgebra structure. The rigidity theorem guarantees that the reductions provided will fit the bill which is one of the most tricky challenge when trying to find the "good" reductions.

The co-terplicial structure on **ST** is combinatorially constructed as follows: for $x \in \mathbf{ST}^r_n$ there is a unique way to describe it as $x_1 \times \dots \times x_p$ such that every x_i is irreducible that is to say that there do not exists $u, v \in \mathbf{ST}$ such that $x_i = u \times v$. Suppose $x = \prod_{j_1,\dots,j_{\lambda(x)}} x' = \prod_{j_1,\dots,j_{\lambda(x)}} u_1 \times \dots \times u_q$ where $u_1 \times \dots \times u_q$ is the irreducible decomposition of x', $u_i \in \mathbf{ST}\,{}^{s_i}_{m_i}$. Denote by $U_1 = u_1 \times \dots \times u_{p_1}$ the decomposition $x = \prod_{j_1,\dots,j_{\lambda(x)}} U_1 \times u_{p+1} \times \dots \times u_q$ with $m_1 + \dots m_{p_1-1} + \lambda(x) < j_{\lambda(x)} \le m_1 + \dots m_{p_1} + \lambda(x)$

$$\Delta_{\triangleleft}(x) = \sum_i \prod_{j_1,\dots,j_{\lambda(x)}} (U_1 \times u_{p_1+1} \times \dots \times u_{p_1+i}) \otimes \mathrm{std}(u_{p_1+i+1} \times \dots \times u_q)$$

$$\Delta_{\triangleright}(x) = \sum_i x_1 \times \dots \times x_i \otimes x_{i+1} \times \dots \times x_p$$

$$\Delta_{\triangledown}(x) = \sum_{i,l} \prod_{j_1,\dots,j_i} (u_1 \times \dots \times u_l) \otimes \prod_{j_{i+1},\dots,j_{\lambda(x)}} (u_{l+1} \times \dots \times u_q)$$

where the last sum runs over i, l such that $m_1 + \dots + m_{l-1} < j_i \le m_1 + \dots + m_l$.

The relations are checked as these cooperations are the dual of the terplicial operations.

Proposition 6.1.1. *The Solomon-Tits algebra is endowed with a terplicial^c-terplicial bialgebra structure and a terplicial^c-tridendriform bialgebra structure.*

Proof. The terplicial^c-terplicial mixed distributive relations are verified.

The proof is based on the unique decomposition of an element into $x = u_1^x \times \dots \times u_{w(x)} \times \prod_{j_1,\dots,j_{\lambda(x)}} u_{w(x)+1} \times u_q^x = u_1^x \times \dots \times u_q^x$ and the definition of the terplicial operations which for two elements x and y will only modify the place of the maxima of x and y according to the operation considered.

For example the relation: $\Delta_{\triangledown}(x\triangledown y) = \Delta_{\triangledown}(x)_1 \otimes \Delta_{\triangledown}(x)_2 \triangledown y + \Delta_{\triangleleft}(x)_1 \otimes \Delta_{\triangleleft}(x)_2 \shortmid\!\triangleright y + x\triangledown\Delta_{\triangledown}(y)_1 \otimes \Delta_{\triangledown}(y)_2 + x \triangleleft \Delta_{\triangleright}(y)_1 \otimes \shortmid\!\triangleright(y)_2 + x \otimes y$ is satisfied as the product $\triangledown$ levels the maxima of x and y while keeping the overall structure of $x' \times y'$ with $x = \prod x'$.

The terplicial^c-tridendriform distributive relations are verified. The proof is based on the unique decomposition of an element $x = \prod_{j_1,\dots,j_{\lambda(x)}} u_1^x \times \ldots \times u_q^x$ and the definition of the tridendriform operations which keeps the overall order (in regards to $\times$) of $u_1^x \times \ldots \times u_q^x$ in x and of $u_1^y \times \ldots \times u_q^y$ in y. $\square$

Theorems 5.3.3 and 5.3.6 give as a corolla that:

Proposition 6.1.2. *The algebra of* **ST** *is free as terplicial algebra and free as a tridendriform algebra.*

7 Combinatorial description of the products, coproducts and confluence law on T_∞.

7.1 Path cutting and stuffled paths

7.1.1 Coproducts indexed by a cutting path: Let $T = \vee(t_1, \dots, t_n)$ a tree and let q be a path in T from the root to a leaf, and denotes its edges $(e_1, \dots, e_k)$. We will refer to q as a cutting path and define the cut of a tree through this path q. Intuitively we duplicate the cutting path and rearrange the left hand-side of the cutting path with the cutting path included by moding out unary edges in order to make it into a tree, and do the same on the right hand-side. Thus, giving us the both trees needed for the coproduct.

It is defined as follows : If the path q is the leftmost path of T, namely l_T, define $\Delta_{l_T}(T)$ as:

$$\Delta_{l_T}(T) = \emptyset \otimes T \ .$$

If the path q is the rightmost path of T, namely r_T, define $\Delta_{r_T}(T)$ as:

$$\Delta_{r_T}(q) = T \otimes \emptyset \ .$$

If q is neither the leftmost nor the rightmost path define $\Delta_q(T)$ as:

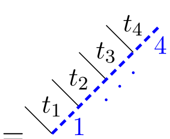
$$\Delta_q(T) = \begin{cases} \Delta_{(e_2,\dots,e_k)}(t_1)_1 \otimes \vee(\Delta_{(e_2,\dots,e_k)}(t_1)_2, t_2, \dots, t_n) \\ \text{ if } e_1 \text{ is the leftmost edge attached to the root} \\ \vee(t_1, \dots, \Delta_{(e_2,\cdots,e_k)}(t_n)_1) \otimes \Delta_{(e_2,\cdots,e_k)}(t_n)_2 \\ \text{ if } e_1 \text{ is the rightmost edge attached to the root} \\ \vee(t_1, \dots, t_{i-1}, \Delta_{(e_2,\dots,e_k)}(t_i)_1) \otimes \vee(\Delta_{(e_2,\dots,e_k)}(t_i)_2, t_{i+1}, \dots, t_n) \\ \text{ if } e_1 \text{is the } i^{\text{th}} \text{ edge attached to the root from left to right.} \end{cases}$$

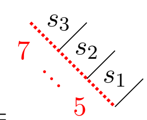
7.1.2 Products indexed by a stuffle path: Consider two trees $T = \vee(t_1, \dots, t_n)$, $S = \vee(s_1, \dots, s_m)$. Consider the rightmost path of T, number its edges $(1, \dots, n)$ ordered from root to leaf, and the leftmost path of S, number its edges $(n+1, \dots, n+m)$ ordered from root to leaf, with n, m integers. Consider the sequence of edges noted (e_p^i) resulting in the inverse image of a $SH(n, m)$ stuffle – recalled in section 6.1– of the edges of the two paths. The sequence will start with 1 or $n+1$ or the set $\{1, n+1\}$, and will be referred to as a *stuffle path* of T and S.

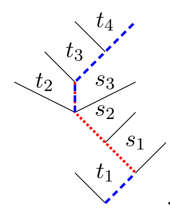
Define the product $T \star_p S$ of two trees accordingly to the stuffle path $p = (e^1, \dots, e^k)$ of T and S inductively as follows:

$$T \star_p S = \begin{cases} \vee(t_1, \dots, t_{n-1}, t_n \star_{(e^2,\dots,e^k)} S) & \text{if } e^1 = 1 \\ \vee(T \star_{(e^2,\dots,e^k)} s_1, s_2, \dots, s_m) & \text{if } e^1 = \{1, n+1\} \\ \vee(t_1, \dots, t_{n-1}, t_n \star_{(e^2,\dots,e^k)} s_1, s_2, \dots, s_m) & \text{if } e^1 = n+1 \end{cases}$$

Example 7.1.1. Consider T and S as represented below where t_i are subtrees of T and s_i are subtrees of S. Denote the edges of the rightmost path of T^* by $(1, \dots 4)$ and the edges of the leftmost path of S by $(5, 6, 7)$. Consider the product of T and S indexed by the stuffle path $p = (1, 5, 6, \{2, 7\}, 3, 4)$. Then,

$T =$ 

, $S =$ 

and $T \star_p S =$ 

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7.2 Combinatorial description of the terplicial structure on T_∞ via stuffle paths and trimming edges. Consider two planar rooted trees T, S, with r_T the rightmost path of T with edges denoted by $(1, \dots, k)$ and l_S the leftmost path of S with edges denoted $(k+1, \dots k+l)$. The terplicial operations defined in Theorem 5.2.2 verify:

$$T \triangleleft S = T \star_{(1,k+1,\dots,k+l,2,\dots,k)} S \ , \qquad T \triangledown S = T \star_{(\{1,k+1\},k+2,\dots,k+l,2,\dots,k)} S \ ,$$
$$T \shortmid\!\triangleright S = T \star_{(k+1,\dots,k+l,1,\dots,k)} S$$

The terplicial cooperations are dual to the above operations and defined by induction with pruning over the edges attached to the root.

7.3 Combinatorial description of the tridendriform operations and cooperations via stuffle paths and cutting paths. For two trees T, S with the edges of the rightmost path of T numbered from 1 to m and the edges of the leftmost path of S numbered from $m+1$ to $m+n$, from root to leaf. We will denote the set of stuffle $\mathcal{P}(T, S)$, the subset of sequences starting with 1 will be denoted $\mathcal{P}_<(T, S)$, the subset starting with $m+1$ will be denoted $\mathcal{P}_>(T, S)$, and the subset of sequences starting with $\{1, m+1\}$ will be denoted $\mathcal{P}_=(T, S)$. The tridendriform operations defined in 5.1.2 are described as follows: Let T and S be two rooted planar trees then

$$T * S = \sum_{p \in \mathcal{P}(T,S)} T \star_p S \ , \quad T \prec S = \sum_{p \in \mathcal{P}_<(T,S)} T \star_p S \ ,$$
$$T \cdot S = \sum_{p \in \mathcal{P}_=(T,S)} T \star_p S \ , \quad T \succ S = \sum_{p \in \mathcal{P}_>(T,S)} T \star_p S$$

It is immediate by induction through description of $T \star_p S$ and the constructive way to define the operation in 5.1.2.

For $p \in SH(n, m)$ one can associate a path $\tilde{p}$ in the product $T \star_p S$ that will be referred to as a *stuffle path*. In the following sequel, we will consider the associated paths instead of the stuffle as we will consider intersections of cutting paths and shuffle paths to describe the confluence laws.

For a tree T denote the set of cutting paths $\mathcal{Q}(T)$, the subset of sequences which start with the leftmost edge of T will be denoted $\mathcal{Q}_>(T)$, the subset of sequence which start with the rightmost edge of T will denoted $\mathcal{Q}_<(T)$, and the subset of sequences which start with neither the rightmost nor the leftmost edge of T will be denoted $\mathcal{Q}_=(T)$.

The co-tridendriform cooperations defined in 5.1.4 are described as follows:

$$\Delta_*(T) = \sum_{q \in \mathcal{Q}(T)} \Delta_q(T) \ , \quad \Delta_\prec(T) = \sum_{q \in \mathcal{Q}_<(T)} \Delta_q(T)$$
$$\Delta_\cdot(T) = \sum_{q \in \mathcal{Q}_=(T)} \Delta_q(T) \ , \quad \Delta_\succ(T) = \sum_{q \in \mathcal{Q}_>(T)} \Delta_q(T)$$

It is immediate by induction considering the description of $\Delta_q(T)$ for a cutting path q and the constructive description of the coproducts in 5.1.4.

Example 7.3.1. Consider Figure 2. For a given tree T, we describe the set of cutting path $\mathcal{Q}(T)$ with the colours to indicate their belonging to the subsets $\mathcal{Q}_>(T)$ if the cutting path is in red, $\mathcal{Q}_=(T)$ if the cutting path is in green, and $\mathcal{Q}_<(T)$ if the cutting path is in blue. Then we give the associated coproducts as sum of Δ_q for a cutting path following the left to right order given in the description of $\mathcal{Q}$.

7.4 Confluence laws of the bialgebra structures on T_∞ In this section, we will investigate the confluence laws on the tridendriform-tridendriform bialgebra structure on T_∞ combinatorically and prove that the coefficients respect the Delannoy series. As in the section 2.2, the coefficients will depend on the cofiltration.

The first step is to compute the confluence laws for the products indexed by a stuffle path with the coproducts indexed by a cutting path.

7.4.1 Confluence law for a product indexed by a stuffle path and a coproduct indexed by a cutting path: Let T and S be two trees. Consider p a stuffle path in $\mathcal{P}(T, S)$, and q a cutting path of $\mathcal{Q}(T \star_p S)$. The intersection between the two paths, $p \cap q$, is a (possibly empty) path in $T \star_p T$ with edges $E(p) \cap E(q)$. Consider also the remaining edges of the cutting path composed of the edges $E(q) \setminus (E(p) \cap E(q))$ and denoted q^c.

Lemma 7.4.1. *q^c is a path, with edges strictly in T or in S or empty.*

We will denote q_T (resp. q_S) when q^c is a path of T (resp. S).

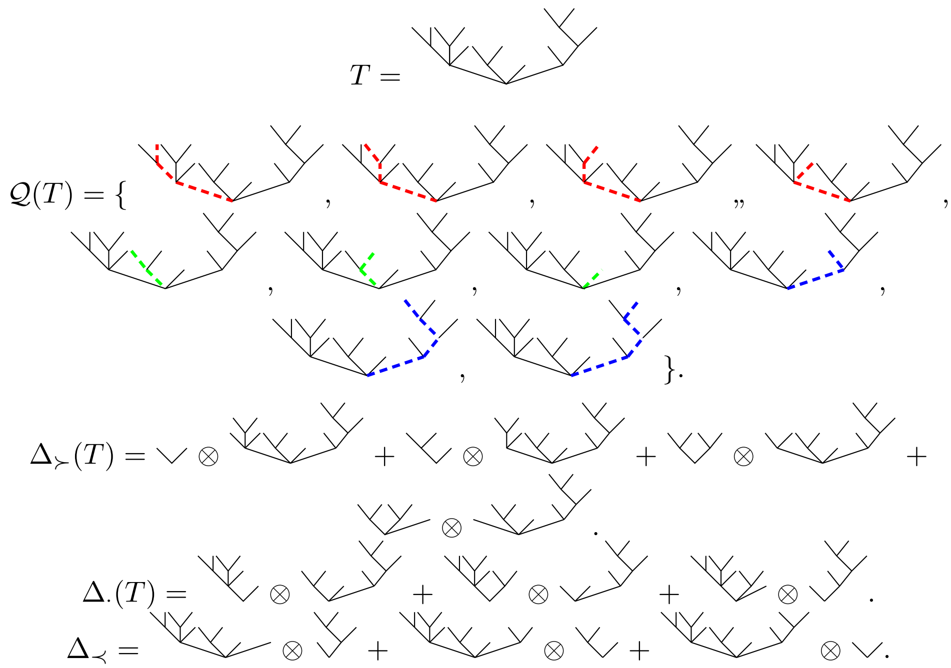


FIGURE 2. $\Delta_{\prec}$, $\Delta_{\cdot}$ et $\Delta_{\succ}$ in terms of thickened paths.

Proof. The proof is analogous to the proof of lemma 2.2.2: it is proven by induction on the number of leaves. It is clear in low dimensions. Suppose the property true for trees such that the sum of their leaves is equal to n. Consider two trees $T = \vee(t_1, \ldots, t_n)$ and $S = \vee(s_1, \ldots, s_m)$ with total number of trees $n+1$. Let the edges of the stuffle path be denoted $(p^i)_{1 \leq i \leq k}$ and the edges of the cutting path $(q^i)_{1 \leq i \leq l}$.

Two cases can occur: $q_1 = p_1$ then consider the stuffle path $(p^2, \ldots, p^k)$ and the cutting path $(q^2, \ldots, q_l)$ in the trees $T^1 = t^n$ and $S^1 = S$ if $p_1 \in E(T) \setminus E(S)$, in the trees $T^1 = T$ and $S^1 = s_1$ if $p_1 \in E(S) \setminus E(T)$ and in the trees $T^1 = t_n$ and $S^1 = s_1$ if $p_1 \in E(T) \cap E(S)$ and conclude by induction.

If $q^1 \neq p^1$ suppose moreover that p^1 is an edge of the rightmost path of T, denoted r_T, but not an edge of the leftmost path of S, denoted l_S. The symmetric case where p^1 is an edge of l_S but not an edge of r_T is analogous. Then by construction, q^1 is an edge attached to the root of T which is not the rightmost edge. Let say it is the edge such that t_o grows from it. So $E(q) \subset E(T) \setminus E(r_T)$ as $\Delta_q(T \star_p S) = \vee(t_1, \ldots, t_{o-1}, \Delta_{(q^2,\ldots,q^l)}(t_o)_1)) \otimes \vee(\Delta_{(q^2,\ldots,q^l)}(t_o)_1), t_{o+1}, \ldots, t_n \star_{p^2,\ldots,p^k} S)$ proving that q is a a path with edges strictly in T. Suppose that p^1 is the first edge of r_T identified to the first edge of l_S, then one concludes by induction on the trees $T^1 = t_n$, $S^1 = s_1$, with the stuffle path $(p^2, \ldots, p^k)$ and cutting path $(q^2, \ldots q^l)$. □

As mentioned above the coproduct of a tree along a cutting path can be understood as duplicating the cutting path to give each side of the tensor. We therefore need to take more notations to describe the confluence law :

When q^c is a path of T we will denote by $[q]_T$ the path in T defined by the sequence of edges $E(q) \cap E(T)$ and $[q]_S$ is defined analougously.

Denote p^c the remaining of the shuffle path in $T \star_p S$, i.e. the path defined by the sequence of edges of $E(p) \setminus (E(p) \cap E(q))$. The intersection path $p \cap q$ has edges in S and in T. We will denote by p_S the sequence of edges $(E(p) \cap E(q) \cap E(S)) \cup E(p^c)$ which is the trace of the shuffle path in S, and analogously defined p_T the sequence of edges $(E(p) \cap E(q) \cap E(T)) \cup E(p^c)$.

Lemma 7.4.2.

$$\Delta_q(T \star_p S) = \begin{cases} T \otimes S & \textit{if } p = q \\ \Delta_{[q]_T}(T)_1 \otimes \Delta_{[q]_T}(T)_2 \star_{p_S} S & \textit{else if } q^c = q_T \\ T \star_{p_S} \Delta_{[q]_S}(S)_1 \otimes, \Delta_{[q]_S}(S)_2 & \textit{else if } q^c = q_S \end{cases}$$

Proof. The proof is analogous to the proof of lemma 2.2.3 and is proven by induction on the number of leaves (for $n = 1, 2, 3$ it is clear). The idea of the proof lies in the fact that the cutting path is determined by a leaf (to which the cutting path leads) and by definition of $T \star_p S$ this leaf is a former leaf of T or of S.

Suppose by induction that the lemma is true for any two tree such that the sum of the number of leaves is less or equal to n. Then consider two planar trees $T = \vee(t_1, \dots, t_n)$ and $S = \vee(s_1, \dots, s_m)$ such that the total number of their leaves equals to $n + 1$. The sequence of edges of the stuffle path p will be denoted $(p^i)_{1 \leq i \leq k}$ and the sequence of edges of the cutting path will be denoted $(q^i)_{1 \leq i \leq l}$. Two cases can occur $p^1 = q^1$ or $p^1 \neq q^1$.

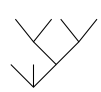
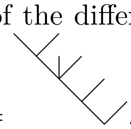
Suppose $p^1 = q^1$, suppose moreover that $p^1 \in E(T) \setminus E(S)$ (the case $p^1 \in E(S) \setminus E(T)$ is symmetrical, the case $p^1 \in E(S) \cap E(T)$ gives the same results choosing either option $p^1 \in E(T)$ or $p^1 \in E(S)$). The definition of the product linked to a stuffle path, and the coproduct linked to a cutting path leads to:

$$\Delta_q(T \star_p S) = \vee(t_1, \dots, t_{n-1}, \Delta_{(q^2,\dots,q^l)}(t_n \star_{(p^2,\dots,p^k)} S)_1) \otimes \Delta_{(q^2,\dots,q^l)}(t_n \star_{(p^2,\dots,p^k)} S)_2 \ .$$

Suppose that $p^1 \in E(T) \cap E(S)$ then

$$\begin{aligned} \Delta_q(T \star_p S) = \vee(t_1, \dots, t_{n-1}, \Delta_{(q^2,\dots,q^l)}(t_n \star_{(p^2,\dots,p^k)} s_1)_1) \otimes \\ \vee(\Delta_{(q^2,\dots,q^l)}(t_n \star_{(p^2,\dots,p^k)} s_1)_2, s_2, \dots, s_n) \ . \end{aligned}$$

The case $p^1 \neq q^1$ is analogous. $\square$

Example 7.4.3. To ease the comprehension of the different paths here is an example of those. Take $T =$  and $S =$ . Consider $T \star_p S$, with p being the stuffle path given by $p = (\{e_1^T, e_1^S\}, e_2^S, e_2^T, e_3^S, \{e_3^T, e_4^S\})$ represented on the right picture of Figure 3 by the dashed blue path. Consider

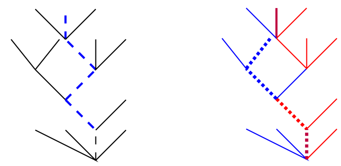


FIGURE 3. Stuffle and cutting path in a product. $T *^p S$

the cutting path q in $T \star_p S$ represented in the figure as a dotted path. The edges from T are in blue, the edges from S are in red and edges from both, i.e. identified edges, are in purple.

7.4.2 Combinatorial description of confluence laws in $\mathbf{T}_\infty$*:* We first prove that the operations and cooperation constructed in 5.2.2 verify the terplicial relations. The terplicial relations are a consequence of the choice of a stuffle path from a path and a stuffle path and their respective first edges.

For example consider three trees T, S, U denote by $(r_i^T)_{1\leq i\leq t_r}, (r_i^S)_{1\leq i\leq s_r}, (r_i^U)_{1\leq i\leq u_r}$ the edges of their rightmost path, and by $(l_i^T)_{1\leq i\leq t_l}, (l_i^S)_{1\leq i\leq s_l}, (l_i^U)_{1\leq i\leq u_l}$ the edges of their leftmost path. Then,

$$\begin{aligned}
(T \shortmid\!\triangleright S) \triangleleft U &= (T \star_{(l_1^S,\dots,l_l^S,r_1^T,\dots,r_{t_r}^T)} S) \star_{(r_1^S,l_1^U,\dots,l_{u_l}^U,r_2^S,\dots,r_{s_r}^S)} U \\
&= T \star_{(l_1^S,\dots,l_l^S,r_1^T,\dots,r_{t_r}^T)} (S \star_{(r_1^S,l_1^U,\dots,l_{u_l}^U,r_2^S,\dots,r_{s_r}^S)} U) \\
&= T \shortmid\!\triangleright (S \triangleleft U) \ .
\end{aligned}$$

The confluence laws given in definition 5.3.1 of coterplicial-terplicial bialgebra and in definition 5.3.4 of coterplicial tridendriform algebras are deduced from lemma 7.4.2 considering the definitions of the products and coproducts given in 7.2 and 7.3.

7.5 Combinatorial confluence laws for the co-tridendriform tridendriform bialgebra structure on $\mathbf{T}_\infty$: A corollary to the above lemma 7.4.2 is that we can count the number of terms of $T \otimes S$ appearing in coproduct of product in the cotridendriform-tridendrifom bialgebra structure of T_∞.

Consider r_T the rightmost path of T and R_T its number of edges denoted $(e_i^T)_i$. Respectively denote L_T to be the number of edges of the leftmost path l_T and denote $(e_i^S)_i$ its edges. We will denote by $D(n,m)$ the Delannoy number of n, m, [24, A266213].

Corollary 7.5.1. *The number of terms of $T \otimes S$ appearing in $\Delta_*(T * S)$ is $D(R_T, L_S)$.*

As a consequence the number of elements $T\otimes S$ in $\Delta_\prec(T \prec S)$ is $D(R_T, L_S - 1)$, in $\Delta_\cdot(T \cdot S)$ is $D(R_T - 1, L_S - 1)$ and in $\Delta_\succ(T \succ S)$ is $D(R_T - 1, L_S)$.

It is the number of distinct terms appearing in $T \prec S, T \cdot S, T \succ S$ respectively.

Proof. The proof lies heavily on the precedent lemma 7.4.2: the element $T \otimes S$ will appear when the cutting path and the stuffle path will coincide. Therefore, it suffices to compute the number of elements of $\mathcal{Q}(T, S)$ which is the collection of stuffles of two collections of cardinal R_T and L_S. We will show that it is $D(R_T, L_S)$ by induction on the cardinal R_T and LS.

In low dimensions, it is obvious. Suppose the property true for $R_T < n, L_S < m$ for any $T, S \in T_\infty$ and consider two trees T, with $R_T = n$, S with $L_S = m$. The computation of $T * S$ when $T = \vee(t_1, \dots, t_n)$, $S = \vee(s_1, \dots, s_m)$ gives through the constructive definitions 5.1.2: $T * S = \vee(T * s_1, s_2 \dots, s_m) + \vee(t_1, \dots, t_{n-1}, t_n * s_1, s_2, \dots, s_m) + \vee(t_1, \dots, t_{n-1}, t_n * S)$. It gives rise to the following inductive property $D(R_T, L_S) = D(R_T, L_S - 1) + D(R_T - 1, L_S - 1) + D(R_T - 1, L_S)$ which is exactly the defining recurrence relation for the Delannoy numbers. □

Proposition 7.5.2. *The edges of r_T are denoted $(e_i^T)_{1 \le i \le R_T}$, the edges of l_S are denoted $(e_i^S)_{1 \le i \le l_S}$.*

Denote by q_i^T the cutting path with $0 \le i \le R_T$ is the maximal integer such that the first i edges are of r_T, i.e. $q_i^T = (e_1^T, \dots, e_i^T, q_{i+1}, \dots e_{|q|})$. Analogously denote q_i^S the cutting path starting with i edges of l_S.

Denote by p_j^T the stuffle path with j the maximal integer such that the first j edges are edges of r_T exclusively. Analogously for p_j^S.

The confluence laws on T_∞ are given by:

$$\begin{aligned}\Delta_\succ(T \succ S) = {} & D(R_T, L_S - 1)\ T \otimes S \\ & + \sum_{\substack{q \in \mathcal{Q}(T) \\ q = q_i^T}} \sum_{\substack{p \in \mathcal{P}_>(T^{e_i^T}, S) \\ p = p_j^S}} D(i, j-1)\ \Delta_q(T)_1 \otimes \Delta_q(T)_2 \star_p S \\ & + \sum_{\substack{q \in \mathcal{Q}_>(S) \\ q = q_i^S}} \sum_{\substack{p \in \mathcal{P}(T, S^{e_i^T}) \\ p = p_j^T}} D(i, j-1)\ T \star_p \Delta_q(S)_1 \otimes \Delta_q(S)_2\end{aligned}$$

$$\begin{aligned}\Delta_\cdot(T \cdot S) = {} & D(R_T - 1, L_S - 1) T \otimes S + \Delta_\cdot(T)_1 \otimes \Delta_\cdot(T)_2 \cdot S + T \cdot \Delta_\cdot(S)_1 \otimes \Delta_\cdot(S)_2 \\ & + \sum_{\substack{q \in \mathcal{Q}_<(T) \\ q = q_i^T}} \sum_{\substack{p \in \mathcal{P}(T^{e_i^T}, S) \\ p = p_j^S}} D(i-1, j-1)\ \Delta_q(T)_1 \otimes \Delta_q(T)_2 \star_p S \\ & + \sum_{\substack{q \in \mathcal{Q}_>(S) \\ q = q_i^T}} \sum_{\substack{p \in \mathcal{P}(T, S^{e_i^T}) \\ p = p_j^S}} D(i-1, j-1)\ T \star_p \Delta_q(S)_1 \otimes \Delta_q(S)_2\end{aligned}$$

$$\begin{aligned}\Delta_{\succ}(T \succ S) =& D(R_T - 1, L_S)\ T \otimes S \\ &+ \sum_{\substack{q \in \mathcal{Q}_{<}(T) \\ q = q_i^T}} \sum_{\substack{p \in \mathcal{P}(T^{e_i^T}, S) \\ p = p_j^T}} D(i-1, j)\ \Delta_q(T)_1 \otimes \Delta_q(T)_2 \star_p S \\ &+ \sum_{\substack{q \in \mathcal{Q}(S) \\ q = q_i^S}} \sum_{\substack{p \in \mathcal{P}_{>}(T, S^{e_i^T}) \\ p = p_j^T}} D(i-1, j)\ T \star_p \Delta_q(S)_1 \otimes \Delta_q(S)_2\end{aligned}$$

$$\begin{aligned}\Delta_{\succ}(T \cdot S) =& \Delta_{\succ}(T)_1 \otimes \Delta_{\succ}(T)_2 \cdot S \\ \Delta_{\succ}(T \prec S) =& \Delta_{\succ}(T)_1 \otimes \Delta_{\succ}(T)_2 \prec S \\ \Delta_{.}(T \succ S) =& T \succ \Delta_{.}(S)_1 \otimes \Delta_{.}(S)_2 \\ \Delta_{.}(T \prec S) =& \Delta_{.}(T)_1 \otimes \Delta_{.}(T)_2 \prec S \\ \Delta_{\prec}(T \succ S) =& T \succ \Delta_{\prec}(S)_1 \otimes \Delta_{\prec}(S)_2 \\ \Delta_{\prec}(T \cdot S) =& T \cdot \Delta_{\prec}(S)_1 \otimes \Delta_{\prec}(S)_2\end{aligned}$$

Proof. The description of the tridendriform products through stuffle paths and the co-tridendriform coproducts through cutting paths 7.3 gives

$$\Delta_{\succ}(T \succ S) = \sum_{p \in \mathcal{P}_{>}(T,S)} \sum_{q \in \mathcal{Q}_{>}(T \star_p S)} \Delta_q(T \star_p S)$$

Then apply lemma 7.4.2. The coefficients appear as for a stuffle path p and a cutting path q, consider any stuffle of the edges $(E(p) \cap E(q)) \cap E(T)$ and $(E(p) \cap E(q)) \cap E(S)$. Consider any path p' obtained from p, and any cutting path q' obtained from q, such that only the edges $(E(p) \cap E(q)) \cap E(T)$ and $(E(p) \cap E(q)) \cap E(S)$ are stuffled, then, $\Delta_{q'}(T \star_{p'} S)$ will give the same element as $\Delta_q(T_p S)$. □

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